UPTEC IT05 037

Examensarbete 20 p
December 2005

Breaking Symmetries in Matrix Models

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Abstract

Breaking Symmetries in Matrix Models

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A number of problems may be considered as constraint satisfaction problems. Such a problem is basically a number of variables that are allowed certain values subject to a number of constraints. An example from real life is the Sudoku puzzles. This paper mainly focus on constraint satisfaction problems formulated with matrix models and how to reduce symmetry in them by adding constraints. In particular a special kind of constraints has been studied, namely lexicographic constraints and a way of simplifying them has been developed. The fully simplified lexicographical constraints for matrix models of size 2x3, 4x3 and 4x4 have been studied. Earlier only the fully simplified lexicographical constraints for the 2x3 matrix had been studied. Minimized conjunctive normal form and disjunctive normal form of the constraints has also been examined. A method for finding a subset of the lexicographical constraints which breaks a major part of the symmetry has also been devised. The results in this paper mainly builds upon earlier research by Flener and Pearson at the ASTRA research group, Uppsala University, and Frich and Harvey at the University of York.

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ISSN: 1401-5749, UPTEC IT05 037

Sammanfattning

Många riktiga problem inom datorvärlden går att betrakta som olika slags optimeringsproblem. Ett optimeringsproblem är som det låter: man försöker optimera något med hänsyn till något annat. Detta låter kanske något abstrakt och för att göra det mer konkret så kan man tänka på olika "väghittar"-tjänster. Dessa är oftast uppbyggda så att givet en start- och en slutpunkt så ska man hitta en väg som är optimerad med avseende på exempelvis sträckan som den resande ska åka. Constraint satisfaction problems, villkorsproblem, är en typ av optimeringsproblem. Det som skiljer dem åt är att specifikationen för ett villkorsproblem gör det möjligt att använda sig av en del generella problemlösningsrutiner för att lösa problemet medan man i optimeringsproblem oftast måste använda problemspecifika lösningsmetoder.

För att få en mer intuitiv bild av vad ett villkorsproblem är, så kan man ta ett klassiskt problem som brukar kallas för 8-damers problemet. Problemet går ut på att placera 8 stycken damer på ett schackbräde. Detta låter sig lätt göras, nästan lite för lätt och det finns därför en del *villkor* som måste vara uppfyllda. Dessa kan formuleras lite olika men går i princip ut på att ingen dam får stå i en position där den kan slå en annan dam. Detta låter sig inte lika lätt göras.

Ett annat exempel är olika sorter av kryptoaritmetiska pussel. Det här prob-

```
   SEND
 + MORE
-------
  MONEY
```

lemet bygger på att varje bokstav skall anta ett heltalsvärde mellan 0 och 9 på så sätt att ekvationen är uppfylld. För att göra det hela lite svårare så måste alla bokstäverna ha olika värden, vilket gör att lösningen där alla bokstäver antar värdet noll inte är giltig.

De ovan nämnda problemen skulle man med fördel kunna formulera som villkorsproblem och använda en villkorsproblemlösare för att lösa. Det sista problemets villkor skulle se ut ungefär som det följande: $1000 \cdot (S + M) + 100 \cdot (E + O) + 10 \cdot (N + R) + (D + E) = 10000 \cdot M + 1000 \cdot O + 100 \cdot N + 10 \cdot E + Y$. Detta ihop med en specifikation av vilka värden som är möjliga för bokstäverna som ingår i uttrycket samt ett villkor som säger att alla bokstäverna skall anta olika värden är allt som behöver anges för att villkorsproblemlösaren skall hitta svaret till problemet.

Tråkigt nog är inte alla problem lika lättlösta som problemen ovan och det tar ibland alltför lång tid för villkorsproblemlösaren att hitta en lösning som uppfyller alla de givna villkoren. En av anledningarna till att det kan ta alltför lång tid är något som brukar kallas för symmetrier. Lösningar som är symmetriska är i det här fallet att betrakta som likadana som någon annan lösning, och om man har hittat den ena lösningen så vill man inte att villkorsproblemlösaren skall tillbringa någon tid med att utforska lösningar som man anser är likadana. För att få en bättre bild av varför man betraktar vissa lösningar som likadana kan vi återgå till exemplet med damerna och schackbrädet. Det är ett problem som har ett antal olika lösningar och ett antal symmetriska lösningar - första gången som man försöker lösa det så känns det inte så, utan man kan lätt inbilla

sig att det inte finns någon lösning! Låt oss anta att vi har hittat en lösning. Om man då vrider hela schackbrädet ett kvartsvarv till höger så har man en ny lösning som uppfyller alla villkoren. Denna lösning brukar man betrakta som symmetrisk med den tidigare lösningen, det vill säga som i någon mening samma lösning.

Det som behandlas i det här arbetet är villkorsproblem representerade med hjälp av matrismodeller. En matrismodell är i det här fallet ett villkorsproblem som innehåller en matris av beslutsvariabler. Beslutsvariabler skulle i exemplet med `SEND+MORE=MONEY` vara de ingående bokstäverna som kan anta olika värden. Den typen av symmetrier som har reducerats är olika kombinationer av rad- och kolumnsymmetrier. En radsymmetri uppkommer när man låter två rader i matrisen byta plats med varandra och en kolumnsymmetri uppkommer genom att låta två kolumner byta plats med varandra. Dessa två sätt kan sedan kombineras med varandra och ge en massa symmetrier. Ett sätt att reducera de symmetrier som uppkommer är genom att till det ursprungliga problemet lägga till extra villkor. Ett populärt sådant villkor är att de olika raderna och kolumnerna måste vara lexikografiskt ordnade. Med lexikografiskt ordnade menar man att om raderna stått i en ordbok så hade en lexikografiskt mindre rad stått före de lexikografiskt större raderna. Det finns mer formella sätt att beskriva detta men då får man titta mer i arbetet. Problemet med rad- och kolumnsymmetrier är tyvärr inte till fullo löst genom att man lägger till villkoren att de olika raderna och kolumnerna skall vara lexikografiskt ordnade. Detta för att det existerar matriser som är symmetriska med varandra men ändå uppfyller villkoren.

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Den övre raden skulle, i en ordbok, stå före den undre raden i båda dessa matriser och de är alltså lexikografiskt ordnade. På samma sätt är även kolumnerna ordnade från vänster till höger. Trots detta är matriserna faktiskt symmetriska. Det kan man inse genom att utgå från matrisen till vänster, byta plats på raderna och i den matris som man får då byta plats på första och sista kolumnen.

Arbetet behandlar vidare olika alternativa villkor, vilka har en nackdel i och med att det krävs ett stort antal av dem för att bryta samtliga symmetrier. Här behandlas också hur man kan minska ner deras antal, göra dem kortare och samtidigt bryta alla rad- och kolumnsymmetrier i matrismodellen. Metoder som inte bryter samtliga symmetrier men dock fler än att ordna raderna och kolumnerna lexikografiskt har också studerats. Slutresultatet är i princip att de metoder som inte bryter samtliga men ändå en stor del av matriserna är att föredra. Detta då det ger en god avvägning mellan antalet villkor som behöver tillföras problemet och hur många symmetrier som bryts.

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Chapter 1

Introduction

Many of the real world problems of today can be considered as optimization problems of some sort. They occur in a lot of different situations, such as banking, logistics and scheduling. Constraint satisfaction problems are a kind of optimization problems and have the possibility to use general heuristics instead of problem specific heuristics for solving the problem. Advantages with using a constraint satisfaction approach include that a fairly large number of problems can be specified in an easy and intuitive way. Consider as an example the simple cryptarithmetic-puzzle in Figure 1.1 in which each of the letters should

```
   SEND
 + MORE
-------
  MONEY
```

Figure 1.1: Simple Cryptarithmetic-Puzzle

be replaced with an integer from zero to nine but a different integer for each different letter. That is, however, not all. The idea is that the numbers one gets when replacing all the letters in SEND and MORE with digits should add up to the number one gets when replacing all the letters in MONEY. This simple problem, easy by hand, might still result in quite a complicated program in an imperative language, such as C or Java. In order to solve this problem in a constraint satisfaction solver, one instructs the program what variables are part of the problem, in this case (in lexicographic order) D, E, M, N, O, R, S, Y, and what values it is possible that they have, in this case any integer between zero and nine. It is also necessary to add the different constraints in some way, the constraints we have from the problem formulation are that all the letters should be different and that SEND+MORE=MONEY. The later constraint is easily expressed as $1000 \cdot (S + M) + 100 \cdot (E + O) + 10 \cdot (N + R) + (D + E) = 10000 \cdot M + 1000 \cdot O + 100 \cdot N + 10 \cdot E + Y$, and most constraint solvers include a built-in predicate for stating that a number of variables have to have different values. The ease with which relatively complicated problems can be stated makes it possible for the programmer to focus on how to solve the problem fast and find out if the solution works.

1.1 Problem Area and Research Questions

Some of the problems that may be formulated as constraint satisfaction problems still remains rather difficult to solve, with respect to limited time and memory. One reason for this is that the problem has a lot of symmetric solutions. A symmetric solution can somewhat simplified be said to be a solution which essentially can be considered as equivalent with at least one other solution. The reason that problems with a lot of symmetries may be harder to solve than problems without are that there is a risk that the constraint solver spends to much time exploring possible solutions which are essentially the same as an already found solution. This problem may be resolved in a few different ways, either by adding extra constraints which break the symmetries or by identifying and removing the symmetries during search. Both of these approaches have been studied, for the first approach see [11, 5, 9] and for the second [2, 6, 10, 12]

The research in this paper mostly builds uppon earlier research aimed at the first of these two methods and obstacles and problems which has arisen there.

The questions that are studied include:

- Is it possible to find a polynomial subset of constraints which breaks most of the symmetries? This question is of interest according to [9] and the reason for this is that the set of constraints which breaks all of the symmetries are huge in the size of the matrix. A method to mechanize the simplifications of the constraints are also needed since there yet is none, see [9, 7, 11].
- Generate and study a set of lexicographical constraints for larger matrices than the earlier studied largest matrix, which is $M_{3\times2}$. Motivation for this research question is found in [11].
- Examine possible domain specific simplifications, both by removing lexicograpic constraints which are redundant when the domain of the decision variables are small and by simplify the logical expression which is equivalent with the constraints.

Purpose

The aim of the paper is to develop more effective and faster ways to solve problems which contain a lot of symmetries. This is not a new area of research and the purpose will be achived by finding answers to some earlier questions that has arisen in the area of constraint logic programming. The specific questions that are considered are listed above. New questions that has arisen during the resarch will be adressed or will be commented upon as possible future directions for research.

1.2 Delimitations

This paper will only consider symmetries in matrix models, and of the possible symmetries only row and column symmetries will be considered. This means that for example rotational symmetries not will be treated. The reason for this delimitation is lack of time and that most earlier research has been conducted on row and column symmetries.

Note that not all possible domains for the decision variables will be considered for all constraints. This is partly due to that domain two is quite much easier to represent in tools used for logic minimization (such as Espresso). However, some tests for domain three will be conducted in order to find out how sensitive the different constraints are for changes in the size of the domain.

Chapter 2

Theory

In this chapter, the theory that is needed for the reader to understand the problem in depth is developed. The discussed theory includes the definition of constraint satisfaction problems, the definition of different kinds of symmetries, some first order predicate logic, and how to minimize different kinds of logic formulas. Most of the theories on how to simplify different kinds of constraints include an example for the case of a 2×3-matrix. This matrix has been choosen because it is the smallest matrix studied in this paper and is illustrative.

2.1 Constraint Satisfaction Problems

In order to reason about constraint satisfaction problems in a more precise way, a formal definition of such problems is needed.

Definition 2.1.1. A **C**onstraint **S**atisfaction **P**roblem, from now on abbreviated as CSP, is a set $S = \{X_1, X_2, \ldots, X_n\}$ of variables and a set $C = \{C_1, C_2, \ldots, C_n\}$ of constraints. Each of the variables $X_i \in S$ is associated with a non empty domain D_i. A constraint $C_j \in C$ specifies a number of variables, belonging to S, and allowable values for them [15]. The variables are often called *decision variables*.

A *state* of a constraint satisfaction problem is an assignment of values to a subset of the variables. A *consistent* assignment is an assignment which does not violate any constraints. If all the variables in a constraint satisfaction problem are assigned a value it is said to be a *complete assignment*. A complete and consistent assignment is called a *solution* to the constraint satisfaction problem [15].

One way to formulate a constraint satisfaction problem in an efficient manner is to use a matrix model for it. A matrix model is a constraint program that contains one or more matrices of decision variables [8]. Some constraint satisfaction problems naturally lend them selves to such a formulation and others are harder to formulate. It is easy to see that it is possible to rewrite every constraint satisfaction problem to include a matrix model, for example by representing the decision variables in the problem as 1×1-matrices, which of course is not a very efficient formulation. However, a lot of the problems that are relatively difficult to formulate as matrices of decision variables can be effectively represented and solved as such [8]. An example of a class of problems that is easy and natural to formulate with a matrice of decision variables are the round robin tournaments (see problem 026 in CSPlib [4]). The problem is in short to schedule a tournament with n teams over $n-1$ weeks, where each week is divided into $n/2$ periods and each period is divided into two slots. Every team takes up one slot when playing. A tournament must satisfy the following three constraints:

- Every team plays once a week.
- Every team plays at most twice in the same period over the tournament.
- Every team plays every other team.

When trying to find a solution to this problem with a naive approach a lot of similar solutions are found. For example, consider the schedule in Table 2.1. From this solution it is easy to find another solution simply by exchanging one

Table 2.1: Simple Round Robin Tournament for n=4

	week 1	week 2	week 3
period 1	A-B	A-C	A-D
period 2	C-D	B-D	B-C

of the columns representing a week with another column representing a different week. Similarly for the rows (periods), and for the two teams of a game. Those solutions may in some sense be considered to be equal, and a program that does not exclude such solutions may acctualy fail to find a correct answer in time.

The idea behind using a CSP-approach for solving such problems is that one does not in detail have to program how different values are assigned to variables, how to implement different constraints and similar things. This is instead considered by a *constraint logic program* (clp) solver. As an example consider the solver provided in SICStus prolog, and the simple cryptarithmetic-puzzle in Figure 1.1.

Example The first row in the program is used for importing the clp for finite domains into the Prolog session and have to be included if the program uses this solver.

The programing of the cryptarithmetic-puzzle is carried out in three different steps:

1. In this step the domain of the different decision variables is stated. In this case all the variables have the same domain, that is $\{0, 1, \ldots, 9\}$.
2. In the second step the different constraints are posted. The constraints are: S and M have to be larger than 0;[1] all of the variables have to be different, and the predicate sum\1 is called. This predicate contains the constraint that $1000 \cdot S + 100 \cdot E + 10 \cdot N + D + 1000 \cdot M + 100 \cdot O + 10 \cdot R + E = 10000 \cdot M + 1000 \cdot O + 100 \cdot N + 10 \cdot E + Y$

[1] This constraint is somewhat consealed in the problem formulation. The reason for it is because integers do not inlude initial zeros.

3. The last step is concerned with what variables the clp solver is going to try to find values for and in what way this is conducted. In SICStus a number of different ways are available, see [3].

The complete program is described below:

```
:- use_module(library(clpfd)).

scrypt([S,E,N,D,M,O,R,Y], Type) :-
     domain([S,E,N,D,M,O,R,Y], 0, 9),      % step 1
     S#>0, M#>0,
     all_different([S,E,N,D,M,O,R,Y]),     % step 2
     sum(S,E,N,D,M,O,R,Y),
     labeling(Type, [S,E,N,D,M,O,R,Y]).    % step 3

sum(S, E, N, D, M, O, R, Y) :-
                  1000*S + 100*E + 10*N + D
     +            1000*M + 100*O + 10*R + E
     #= 10000*M + 1000*O + 100*N + 10*E + Y.
```

A CSP can also be considered as a standard search problem. Such a problem consists of an initial state, a successor function, a goal test, and a path cost. In the case of a CSP the initial state is when all the variables are unassigned, the successor function is any variable assignment which does not conflict with an earlier variable assignment, the goal test checks if the variable assignment is complete and finally, the path cost is a constant. As earlier mentioned there is different techniques for choosing what variable assignment to do. The default in SICStus is leftmost, which means that the leftmost variable is choosen for assignment. A common search strategy used for CSPs are depth first-search algorithms. The reason for this is that if the problem involves n variables the solution to the problem has to be found at depth n in the tree since a solution has to be a complete assignment. For a more complete treatment of different search strategies and methods for assigning values to variables see [15].

2.2 Logic

Constraints are logical formulas. A brief introduction to the area of first-order predicate logic is therefore presented in this section.

This is only a brief introduction to predicate logic and the interested reader is referenced to Nerode and Shore [13] for a more complete introduction. First, the different kind of symbols which are allowed in a logic expression is defined.

Definition 2.2.1. A language L consists of the following sets of symbols:

1. Variables: $x, y, z, v, x_0, x_1, \ldots, y_0, y_1, \ldots$
2. Constants: $c, d, c_0, d_0, \ldots$
3. Connectives: $\wedge, \neg, \vee, \rightarrow, \leftrightarrow$
4. Quantifiers: $\exists, \forall$
5. Predicate symbols: $P, Q, R, P_0, P_1, \ldots, R_0, R_1 \ldots$

6. Function symbols: $f, g, h, f_0, f_1, \ldots, g_0, \ldots$ of different arities.
7. Punctation: the comma , and the left and right parantheses (,).

Definition 2.2.2. A term is:

1. Every variable is a term
2. Every constant is a term
3. If f is an n-ary function symbol, $n \in \mathbb{N}$, and $t_1, t_2, \ldots, t_n$ are terms, then $f(t_1, \ldots, t_n)$ is a term.

Definition 2.2.3. An atomic formula is $R(t_1, t_2, \ldots, t_n)$ where R is an n-ary predicate symbol and $t_1, t_2, \ldots, t_n$ are terms.

Definition 2.2.4. The following are formulas:

1. Every atomic formula is a formula.
2. If α and β are formulas then so are $(\alpha \wedge \beta), (\neg\alpha), (\alpha \vee \beta), (\alpha \to \beta)$ and $(\alpha \leftrightarrow \beta)$.
3. If v is a variable and α is a formula, then $((\exists v)\alpha)$ and $((\forall v)\alpha)$ are formulas.

Definition 2.2.5. Subformula and open formula.

1. If α is a formula and β is a consecutive sequence of symbols from α and also a formula, then β is a subformula of α.
2. An occurence of a variable v in a formula φ is bound if there is a subformula ψ of φ containing that occurence of v such that ψ begins with $((\forall)$ or $((\exists)$. (This includes the v in $\forall v$ and $\exists v$ that are bound by this definition.) An occurence of v in φ is *free* if it is not bound.
3. A variable v is said to *occur free* in φ if it has at least one free occurrence there.
4. An open formula is a formula with no quantifiers[2].

Theorem 2.2.6. Prenex Normal Form
For every formula α there exists an equivalent formula β with the same free variables in which all quantifiers appear at the beginning. β is called a prenex normal form *of α.*

Proof. Omitted, see [13], page 129. □

Definition 2.2.7. A conjunctive normal form (CNF) of a formula $B\alpha$ is a formula $B(\alpha_{1,1} \vee \alpha_{1,2} \vee \ldots \vee \alpha_{1,n_1}) \wedge (\alpha_{2,1} \vee \alpha_{2,2} \vee \ldots \vee \alpha_{2,n_2}) \wedge \ldots \wedge (\alpha_{k,1} \vee \alpha_{k,2} \vee \ldots \vee \alpha_{k,n_k})$, where $\alpha_{1,1}, \alpha_{1,2}, \ldots, \alpha_{1,n_1}, \alpha_{2,1}, \alpha_{2,2}, \ldots, \alpha_{2,n_2}, \ldots, \alpha_{k,1}, \alpha_{k,2}, \ldots, \alpha_{k,n_k}$ are atomic formulas and B is the consecutive sequence of quantifiers in the prenex normal form of $B\alpha$.

[2]Normally an open formula has at least one free variable but this definition is in accordance with [13] and has therefore been used.

Definition 2.2.8. A disjunctive normal form (DNF) of a formula $B\alpha$ is a formula $B(\alpha_{1,1} \wedge \alpha_{1,2} \wedge \ldots \wedge \alpha_{1,n_1}) \vee (\alpha_{2,1} \wedge \alpha_{2,2} \wedge \ldots \wedge \alpha_{2,n_2}) \vee \ldots \vee (\alpha_{k,1} \wedge \alpha_{k,2} \wedge \ldots \wedge \alpha_{k,n_k})$, where $\alpha_{1,1}, \alpha_{1,2}, \ldots, \alpha_{1,n_1}, \alpha_{2,1}, \alpha_{2,2}, \ldots, \alpha_{2,n_2}, \ldots, \alpha_{k,1}, \alpha_{k,2}, \ldots, \alpha_{k,n_k}$ are atomic formulas and B is the consecutive sequence of quantifiers in the prenex normal form of $B\alpha$.

A *litteral* is in this context the same as an atomic formula or its negation. In this paper only formulas without any quantifiers will be considered so B in the above definitions will always be of length 0.

Definition 2.2.9. Let $[x_1, x_2, \ldots, x_n] \leq_{lex} [y_1, y_2, \ldots, y_n]$ be defined to be $(x_1 < y_1) \wedge (x_1 = y_1 \rightarrow x_2 \leq y_2) \wedge (x_1 = y_1 \wedge x_2 = y_2 \rightarrow x_3 \leq y_3) \wedge \ldots$. An alternative recursive definition is: Let $[x_1, x_2, \ldots, x_n]$ and $[y_1, y_2, \ldots, y_n]$ be two sequences of values. Then $[x_1, x_2, \ldots, x_n] \leq_{lex} [y_1, y_2, \ldots, y_n]$ is:

- For any two sequences $[x_i], [y_i]$, of length 1, $[x_i] \leq_{lex} [y_i]$ if $x_i = y_i$ or $x_i < y_i$
- For any two sequences $[x_1, x_2, \ldots, x_i], [y_1, y_2, \ldots, y_i]$, of length greater than 1, $[x_1, x_2, \ldots, x_i] \leq_{lex} [y_1, y_2, \ldots, y_i]$ if $x_1 < y_1$ or if both $x_1 = y_1$ and $[x_2, \ldots, x_n] \leq_{lex} [y_2, \ldots, y_n]$ are true.

Definition 2.2.10. De Morgan's Laws

1. $\neg(\alpha \vee \beta) \leftrightarrow (\neg\alpha \wedge \neg\beta)$
2. $\neg(\alpha \wedge \beta) \leftrightarrow (\neg\alpha \vee \neg\beta)$

De Morgan's Laws also exist in a more generalised version.

Theorem 2.2.11. *De Morgan's Law, generalised version* $\neg(\alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_n) \leftrightarrow \neg\alpha_1 \wedge \neg\alpha_2 \wedge \ldots \wedge \neg\alpha_n$*, where* $\alpha_1, \alpha_2, \ldots, \alpha_n$ *are atomic formulas.*

Proof. By induction, base case: $\neg(\alpha_1) \leftrightarrow \neg\alpha_1$, obvious true. $\neg(\alpha_1 \vee \alpha_2) \leftrightarrow (\neg\alpha_1 \wedge \neg\alpha_2)$ true by De Morgan's Laws.Assumption for induction: (i) $\neg(\alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_k) \leftrightarrow \neg\alpha_1 \wedge \neg\alpha_2 \wedge \ldots \wedge \neg\alpha_k$, we want to show that (i) implies that $\neg(\alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_{k+1}) \leftrightarrow \neg\alpha_1 \wedge \neg\alpha_2 \wedge \ldots \wedge \neg\alpha_{k+1}$. Let β be $\alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_k$, then is $\neg(\beta \vee \alpha_{k+1}) \leftrightarrow \neg\beta \wedge \neg\alpha_{k+1}$, by De Morgan's Laws. This, however, equals $\neg(\alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_k) \wedge \neg\alpha_{k+1}$, which by the induction hypothesis equals $\neg\alpha_1 \wedge \neg\alpha_2 \wedge \ldots \wedge \neg\alpha_{k+1}$ □

Theorem 2.2.12. $\neg(\alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_n) \leftrightarrow \neg\alpha_1 \vee \neg\alpha_2 \vee \ldots \vee \neg\alpha_n$*, where* $\alpha_1, \alpha_2, \ldots, \alpha_n$ *are atomic formulas.*

Proof. Almost identical with that of Theorem 2.2.11. □

2.3 Symmetries

Symmetries are to a large extent to blame for making some problems almost unsolvable in a practical sense, with limited time and memory. In this section different kinds of symmetries, such as row and column symmetry in matrix models, are defined. Symmetry in a more general context is also defined.

Row symmetry in a matrix can be thought of as allowing the rows to swap place with each other. The matrix before the swap and the matrix after the swap are then said to be row-symmetric. If one instead allows two columns to swap place it is called a column symmetry. A more formal definition is as follows:

Definition 2.3.1. The following are different kinds of symmetries [9]:

- A *symmetry* is a bijection on the decision variables that preserves solutions and non-solutions.
- A *row symmetry* of a 2-d matrix is a bijection between the variables of two of its rows that preserves solutions and non-solutions.
- A *column symmetry* of a 2-d matrix is a bijection between the variables of two of its columns that preserves solutions and non-solutions.

2.4 Breaking Symmetries

This section explains different approaches in order to break all, or most, of the symmetries in CSP. It is, however, primarily concerned with methods and theories for breaking symmetries *in matrix models* of constraint satisfaction problems *by adding constraints*. It is also possible to break the symmetries by modifying the search procedure used and by adding constraints during search, see for example [2, 6], the global cut framework (GCF) [10] or the symmetry-breaking during search framework (SBDS) [12].

2.4.1 Adding Constraints

Lexicographic constraints are a special kind of constraints that can be used for breaking symmetries in matrix models. They are easy to use and have earned a lot of interest, see for example [11, 5, 9].

The lex^2-constraints –fails to remove all the symmetries

Flener et al. [7] has shown that one can consistently add the lexicographic constraint that both the rows and the columns should be lexicographically ordered. This constraint is called lex^2. It was also shown that even though the constraint successfully removes a number of the symmetries it fails to remove them all. This was also independently shown by Shlyakhter [16]. In order to illustrate the use of these constraints consider the matrix consisting of two rows and three columns in Figure 2.1. The constraint that the two rows should be

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \end{bmatrix}$$

Figure 2.1: Matrix for the 2×3-case

lexicographically ordered can be expressed as in formula 2.1:

$$[x_1, x_2, x_3] \leq_{lex} [x_4, x_5, x_6] \quad (2.1)$$

Table 2.2: Elements in the complete symmetry group for $M_{2\times 3}$

Permutation	Name	Largest Cycle
()	id	1
(1,2)(4,5)	$P_{c_{12}}$	2
(2,3)(5,6)	$P_{c_{23}}$	2
(1,4)(2,5)(3,6)	$P_{r_{12}}$	2
(1,6,2,4,3,5)	P_{δ}	6
(1,5,3,4,2,6)	P_{σ}	6
(1,4)(2,6)(3,5)	P_{α_1}	2
(1,5)(2,4)(3,6)	P_{α_2}	2
(1,6)(2,5)(3,4)	P_{α_3}	2
(1,3)(4,6)	$P_{c_{13}}$	2
(1,2,3)(4,5,6)	$P_{c_{123}}$	3
(1,3,2)(4,6,5)	$P_{c_{132}}$	3

The constraints that the three columns should be lexicographically ordered can be expressed as two different constraints[3], see formula 2.2.

$$\begin{aligned} [x_1, x_4] &\leq_{lex} [x_2, x_5] \\ [x_2, x_5] &\leq_{lex} [x_3, x_6] \end{aligned} \tag{2.2}$$

In order to see that those constraints not are enough to break all the symmetries for a 2×3-matrix it is sufficient to consider the following situation:

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

None of these matrices contradicts the lex^2-constraint. They are, however, symmetric with each other as it is possible to get from the left matrix to the right by first swapping the two rows, and then swapping the first column with the last column. In order to break even more symmetries other methods are thus needed.

The *lex*-constraints

A method for breaking all the symmetries in a matrix problem was developed in [5]. The method uses some group theory and a short description of the method follows.

Example Let M be the matrix in Figure 2.1. It can be represented as a vector of length 6, $(x_1, x_2, x_3, x_4, x_5, x_6)$. The different permutations of the elements can in cycle notation be described as $(1,4)(2,5)(3,6)$, $(1,2)(4,5)$ and $(2,3)(5,6)$. Those three generators generate the complete symmetry group for M, consisting of 12 elements, including the identity permutation. The complete symmetry group for M is described in Table 2.2.

[3] In SICStus Prolog it is possible to express them as a single constraint with the use of the predicate lex_chain/1

$$
\begin{array}{ll}
[x_1, x_2, x_3, x_4, x_5, x_6] \leq_{lex} [x_2, x_1, x_3, x_5, x_4, x_6] & (c_{12}) \\
[x_1, x_2, x_3, x_4, x_5, x_6] \leq_{lex} [x_1, x_3, x_2, x_4, x_6, x_5] & (c_{23}) \\
[x_1, x_2, x_3, x_4, x_5, x_6] \leq_{lex} [x_4, x_5, x_6, x_1, x_2, x_3] & (r_{12}) \\
[x_1, x_2, x_3, x_4, x_5, x_6] \leq_{lex} [x_6, x_4, x_5, x_3, x_1, x_2] & (\delta) \\
[x_1, x_2, x_3, x_4, x_5, x_6] \leq_{lex} [x_5, x_6, x_4, x_2, x_3, x_1] & (\sigma) \\
[x_1, x_2, x_3, x_4, x_5, x_6] \leq_{lex} [x_4, x_6, x_5, x_1, x_3, x_2] & (\alpha_1) \\
[x_1, x_2, x_3, x_4, x_5, x_6] \leq_{lex} [x_5, x_4, x_6, x_2, x_1, x_3] & (\alpha_2) \\
[x_1, x_2, x_3, x_4, x_5, x_6] \leq_{lex} [x_6, x_5, x_4, x_3, x_2, x_1] & (\alpha_3) \\
[x_1, x_2, x_3, x_4, x_5, x_6] \leq_{lex} [x_3, x_2, x_1, x_6, x_5, x_4] & (c_{13}) \\
[x_1, x_2, x_3, x_4, x_5, x_6] \leq_{lex} [x_2, x_3, x_1, x_5, x_6, x_4] & (c_{123}) \\
[x_1, x_2, x_3, x_4, x_5, x_6] \leq_{lex} [x_3, x_1, x_2, x_6, x_4, x_5] & (c_{132})
\end{array}
$$

Figure 2.2: Lex-constraints for $M_{2\times 3}$

For a general $M_{m\times n}$-matrix there are a lot of elements in the generated group, actually the number is $m! \cdot n!$ [9]. Those twelve elements in turn result in twelve lexicographic constraints, one for each element in the group, see Figure 2.2. In order of how to interpret the constraints see the definition of $\leq_{lex}$ on page 15.

The number of constraints generated in this way for a general $M_{m\times n}$-matrix is huge for larger matrices, actually the number is $m! \cdot n! - 1$ if the constraint from the identity permutation is not counted [9]. The reason why this constraint ordinarily is not included is because it is always true. There is a GAP routine, written by Justin Pearson, which generates the lexicographic constraints for a matrix given n and m. Further on the actual construction of the lexicographic constraints will not be treated.

In order to get a more manageable set of constraints there has been been a lot of focus on possible ways to simplify them, see section 2.5 and 2.6.

The number of different matrices when both row and column symmetry are considered is given by Polya's theorem and any two matrices are considered to be different if they are not symmetric with each other. This theorem is usefull in veryfing that the correct number of solutions are found. Let D be a finite set of elements, and let A be a set of permutations of the elements of D. Each element in A can be written as a set of cycles since D is finite. Let $v(a,i)$ be the number of cycles with length i in a. Next consider the set of mappings from D to a finite set R, and define an equivalence relation on F by: $f_1 \sim f_2$ if and only if for some $a \in A$ we have$f_1 = f_2 \circ a$. Let F_A denote the set of equivalence classes induced by this operation. Polya's theorem states:

$$|F_A| = \frac{1}{|A|} \sum_{a\in A} \prod_{i=1}^{|D|} |R|^{v(a,i)} \tag{2.3}$$

In this case D is the set of atomic formulas, A the set of symmetries and $R = true, false$. F_A is then the set of different interpretations of the theory and the cardinality of F_A equals the number of distinct matrices. The cardinality of F_A

is then given by:

$$\frac{1}{|A|}\sum_{a\in A}\prod_{i=1}^{|D|}2^{v(a,i)} \tag{2.4}$$

For the 2×3-matrix, with domain size 2, this will give $|F_A| = \frac{1}{12}\cdot(2^6+2^2\cdot 2^2+2^2\cdot 2^2+2^3+2+2+2^3+2^3+2^3+2^2\cdot 2^2+2^2+2^2) = 13$ different matrices.

2.5 Simplifications — Domain Independent

2.5.1 Simplification of *lex*-constraints

Earlier research in the area of symmetry breaking has been conducted by Frisch and Harvey, who used two rules in order to simplify the set of symmetry breaking constraints in a 3×2 matrix [11]. Slightly less powerful rules were discussed in [9]. In this section a new rule which supersedes the rules devised by Frisch and Harvey is presented. The presented rule is strictly stronger in the sense that it simplifies lexicographical constraints which are not simplified by any of the rules by Frisch and Harvey. It is, however, unclear if lexicographical constraints which are not simplified by the Frisch and Harvey rules ever will appear in the set of constraints for matrices of different sizes. The main motivation our (unique) rule as a replacement for their (two) rules was its easier implementation. The notation of the rules by Frisch and Harvey has been slightly modified in order to increase the readability, for the original version see [11].

Rule 1. If we have a constraint C of the form $\alpha X \beta \leq_{lex} \gamma Y \delta$ and $\alpha = \gamma$ logically implies $X = Y$, then we may replace it with $\alpha\beta \leq_{lex} \gamma\delta$.

Rule 1 is only considered with *internal simplifications*, which mean a simplification of one constraint with no regard taken to other constraints.

Rule 2. If we have a set of constraints C of the form $C' \cup \{\alpha\beta \leq_{lex} \gamma\delta\}$, where C' is a set of constraints, and $C' \cup \{\alpha = \gamma\}$ logically implies $\beta \leq_{lex} \delta$, then we may replace C with $C' \cup \{\alpha \leq_{lex} \gamma\}$.

α, β, δ and γ are in this context segments of the lexicographic constraints. Two segments are equal if and only if any position in the first part of the constraint is equal to the corresponding position in the second part of the constraint. Specifically, the segment $[0, 1, 1]$ is not equal to $[1, 1]$. The segments are also allowed to be of length zero. X and Y are variables and may be considered as segments of length one. One significant difference between Rule 2 and Rule 1 is that Rule 2 takes all of the constraints into consideration. To see how Rule 1 works consider the following example:

Example Consider the first constraint in Figure 2.2. The constraint is $[x_1, x_2, x_3, x_4, x_5, x_6] \leq_{lex} [x_2, x_1, x_3, x_5, x_4, x_6]$. Apply rule 1 with $\alpha = [x_1, x_2]$, $\gamma = [x_2, x_1]$, $X = x_3$ and $Y = x_3$. It is trivially true that $X = Y$ is implied and the constraint can hence be simplified to $[x_1, x_2, x_4, x_5, x_6] \leq_{lex} [x_2, x_1, x_5, x_4, x_6]$. In the next step let $\alpha = [x_1]$, $\gamma = [x_2]$, $X = x_2$ and $Y = [x_1]$. Let $\alpha = \gamma$, which results in $x_1 = x_2$ and $X = Y$ is thus implied. The resulting constraint is then $[x_1, x_4, x_5, x_6] \leq_{lex} [x_2, x_5, x_4, x_6]$. Let $\alpha = [x_1, x_4]$, $\gamma = [x_2, x_5]$, $X = x_5$ and $Y = x_4$. $X = Y$ is implied because the assumption that $[x_1, x_4] = [x_2, x_5]$ implies that $x_4 = x_5$. The constraint can then by Rule 1 be simplified to $[x_1, x_4, x_6] \leq_{lex} [x_2, x_5, x_6]$. x_6 in the constraint can also be removed by applying Rule 1 with $\alpha = [x_2, x_5]$, $\gamma = [x_1, x4]$. This results in the constraint $[x_1, x_4] \leq_{lex} [x_2, x_5]$, which not can be further simplified by use of Rule 1. The result of simplifying all the constraints in Figure 2.2 is shown in Figure 2.3.

As one can see the internal simplifications can result in quite significant reductions in the length of the different lexicographic constraints.

$$[x_1, x_4] \leq_{lex} [x_2, x_5] \quad (c_{12})$$
$$[x_2, x_5] \leq_{lex} [x_3, x_6] \quad (c_{23})$$
$$[x_1, x_2, x_3] \leq_{lex} [x_4, x_5, x_6] \quad (r_{12})$$
$$[x_1, x_2, x_3, x_4, x_5] \leq_{lex} [x_6, x_4, x_5, x_3, x_1] \quad (\delta)$$
$$[x_1, x_2, x_3, x_4, x_5] \leq_{lex} [x_5, x_6, x_4, x_2, x_3] \quad (\sigma)$$
$$[x_1, x_2, x_3] \leq_{lex} [x_4, x_6, x_5] \quad (\alpha_1)$$
$$[x_1, x_2, x_3] \leq_{lex} [x_5, x_4, x_6] \quad (\alpha_2)$$
$$[x_1, x_2, x_3] \leq_{lex} [x_6, x_5, x_4] \quad (\alpha_3)$$
$$[x_1, x_4] \leq_{lex} [x_3, x_6] \quad (c_{13})$$
$$[x_1, x_2, x_4, x_5] \leq_{lex} [x_2, x_3, x_5, x_6] \quad (c_{123})$$
$$[x_1, x_2, x_4, x_5] \leq_{lex} [x_3, x_1, x_6, x_4] \quad (c_{132})$$

Figure 2.3: Internal Simplified Constraints for $M_{2\times3}$

It is, however, possible to replace both Rule 1 and Rule 2 with a single rule which is strictly stronger in the sense that it simplifies lexicographical constraints which are not simplified by any combination of Rule 1 and 2. The stronger rule was discovered in the implementation of rules Rule 1, Rule 2 and the rules in [9].

Rule 3. If we have a set of constraints C of the form $C' \cup \{\alpha X \beta \leq_{lex} \gamma Y \delta\}$, where C' is a set of constraints, and $C' \cup \{\alpha = \gamma\}$ logically implies $X = Y$ (or if is the case that $X \leq Y$ is implied and β and δ is of size 0), then we may replace C with $C' \cup \{\alpha\beta \leq_{lex} \gamma\delta\}$.

In this rule both β and δ are allowed to be of length zero. In order to show that Rule 3 may replace both Rule 1 and 2 it is first shown that it supersedes Rule 1.

Theorem 2.5.1. *Let S be a set of constraints and let C_1, C_2 be constraints. If $S \cup \{C_1\}$ can be simplified by Rule 1 into a different set of constraints $S \cup \{C_2\}$, then $S \cup \{C_1\}$ also be simplified by application of Rule 3 to $S \cup \{C_2\}$.*

Proof. Consider a set of constraints $S \cup \{C_1\}$ and let $C_1 = \alpha X \beta \leq_{lex} \gamma Y \delta$. Assume that C_1 can be simplified into $C_2 = \alpha\beta \leq_{lex} \gamma\delta$ by Rule 1. Clearly $S \cup \{C_1\}$ is equivalent to $S \cup \{C_2\}$ since C_1 is equivalent to C_2. From the assumption and Rule 3 it follows that $S \cup \{\alpha X \beta \leq_{lex} \gamma Y \delta\}$ can be simplified into $S \cup \{\alpha\beta \leq_{lex} \gamma\delta\}$, which is exactly the same as $S \cup \{C_2\}$. □

It remains to show that Rule 3 also supersedes Rule 2.

Theorem 2.5.2. *Let S be a set of constraints and let C_1, C_2 be constraints. If $S \cup \{C_1\}$ can be simplified by Rule 2 to a different set of constraints $S \cup \{C_2\}$ it is also possible to simplify $S \cup \{C_1\}$ to $S \cup \{C_2\}$ by repeated application of Rule 3.*

Proof. Consider a set of constraints $S \cup \{\alpha\beta \leq_{lex} \gamma\delta\}$. Assume that $S \cup \{\alpha\beta \leq_{lex} \gamma\delta\}$ can be simplified into $S \cup \{\alpha \leq_{lex} \gamma\}$ by Rule 2. From the assumption and

Rule 2 it follows that $\beta \leq_{lex} \delta$. Since it never can be the case that it is logically implied that any position in β is less than the corresponding position in δ it suffices to consider when they are equal. If we let $\beta = \beta_1 X$ and $\delta = \delta_1 Y$, then X and Y can be dropped by use of Rule 3 and further on the next position to the left and so on resulting in $S \cup \{\alpha \leq_{lex} \gamma\}$. □

The next step is to show that Rule 3 indeed is able to simplify constraints which Rule 1 and Rule 2 are unable to simplify.

Theorem 2.5.3. *Rule 3 is strictly stronger than any combination of Rule 1 and 2 in the sense that it simplifies a set of constraints which is not simplified by any combination of Rule 1 and Rule 2.*

Proof. Consider the set of constraints:

$$[A, B, C] \leq_{lex} [D, E, F] \tag{2.5}$$

$$[B] \leq_{lex} [E]$$

$$[E] \leq_{lex} [B]$$

This set of constraints can be further simplified to:

$$[A, C] \leq_{lex} [D, F] \tag{2.6}$$

$$[B] \leq_{lex} [E]$$

$$[E] \leq_{lex} [B]$$

because $[B] \leq_{lex} [E]$ and $[E] \leq_{lex} [B]$ implies that $E = B$. It is not possible to simplify the set of constraints in 2.5 with Rule 1 or Rule 2 or any combination of them. The reason for this is that Rule 2 only is able to remove the end of a constraint and not an occurrence of a variable inside a constraint. However, consider Rule 3 and let $\alpha = [A]$, $\beta = [C]$, $\delta = [F]$, $\gamma = [D]$ and $S = \{[B] \leq_{lex} [E], [E] \leq_{lex} [B]\}$. It is then possible to simplify the constraints to $S \cup \{[A, C] \leq_{lex} [D, F]\}$. □

The use of Rule 3 on either the constraints in Figure 2.2 or Figure 2.3 results in the same set of constraints.

Example This set of constraints represents the set of constraints after applying Rule 1 and 2 on the constraints in Figure 2.2 or directly applying Rule 3 on the constraints in Figure 2.3.

Table 2.3: Completely Simplified *lex*-constraints, $M_{2\times 3}$

$$\begin{aligned}
[x_1, x_2, x_3] &\leq_{lex} [x_4, x_5, x_6] \\
[x_1, x_2, x_3] &\leq_{lex} [x_6, x_5, x_4] \\
[x_1, x_2, x_3] &\leq_{lex} [x_6, x_4, x_5] \\
[x_1, x_2, x_3] &\leq_{lex} [x_5, x_4, x_6] \\
[x_1, x_2, x_3, x_4] &\leq_{lex} [x_5, x_6, x_4, x_2] \\
[x_1, x_2, x_3] &\leq_{lex} [x_4, x_6, x_5] \\
[x_1, x_4] &\leq_{lex} [x_2, x_5] \\
[x_2, x_5] &\leq_{lex} [x_3, x_6]
\end{aligned}$$

Frisch and Harvey [11] conjecture that there does not exist a set of symmetry breaking constraints which is simpler (having fewer or shorter constraints) and logically equivalent to the set of constraints generated by Rule 1 and Rule 2 in the $M_{2\times3}$-case. Since thoose rules generate the same set of constraints in the $M_{2\times3}$-case as is generated by Rule 3 it follows that if their conjecture is true also Rule 3 generates a minimal set of lexicographic constraints. They also suggest that it might be useful to study a larger matrix and the symmetry breaking constraints for it. In order to simplify the constraints for matrices other than the 2×3 it is preferable to use Rule 3 because it is not known whether constraints that will not be simplified by Rule 1 and Rule 2 will appear.

A remaining question is how to decide if a set of constraints actually implies, for example, $X = Y$ in Rule 3 and how to implement it. How this may be done will be considered in the next section.

Algorithm which Implements Rule 3

Let a lexicographic constraint be represented by a tuple, (α, β), where α is the left side of the constraint and β is the right side of the constraint. Both α and β are lists. Let all the constraints together be represented as a list of lexicographic constraints. The method to simplify them will be as follows:

Select one of the constraints for simplification, presumably the first in the list representing them. For each position in this constraint we will decide whether the position will be included in the simplified constraint or not. If no position will be included, the constraint is completely redundant and will be removed.

In order to decide if a position will be included all earlier positions in the constraint to be simplified, from here on CS, will be assumed equal. For example $[A, B, C] \leq_{lex} [D, E, F]$ and the position under discourse is two, then $A = D$ and $B = E$ will be assumed. If this results in that the variables at the position which is under discussion are equal they can be removed, which is in accordance with Rule 3. The reason for this is that it will only be relevant to look at this position if all earlier variables are equal (in the sense examplified above), and if the result of having all the variables equal always results in that the variables at the position automatically are equal it will never be relevant to actually compare them.

However, it is possible to remove this position even in other instances. It can be removed if the assumption that all earlier positions in CS are equal together with the other constraints in the set of constraints implies that the variables at the position under discussion are equal. The reason for this is that each of the constraints have to be satisfied in order for the set of constraints to be satisfied. The relevant positions to consider in the different constraints depends on in which position each constraint differ. If one of them differ at the first position it is only relevant to consider this position for that particular constraint. Consider as an example $[A, B, C] \leq_{lex} [D, E, F]$ and A is less than D, then the constraint is true irrelevant of the values of B and E. However, if $A = D$ for some reason then it is relevant to consider the next position in the constraint. One reason that A should be equal to D is that we have the constraints $[A] \leq_{lex} [B]$ and $[B] \leq_{lex} [A]$ which implies that $A = B$.

If the position under discussion can not be removed in any of the ways above it exists at least one assignment which will make the constraint true at the position under discussion and at least one assignment which will make the constraint false at the same position, under the assumption that the domain the variables are sufficently large.

This idea will be implemented in the following way: Let SC be the set of constraints under consideration and let $\alpha \leq_{lex} \beta$ be the constraint which will be simplified. Let in a constraint $\alpha \leq_{lex} \beta$, n, $[\alpha^n \leq_{lex} \beta^n]$ be the position under consideration, and start with considering the last position. Then for each i, $i > 0$, $i < n$, add the variables at $\alpha^i \leq_{lex} \beta^i$ as vertices to the directed graph G, if they are not already added. The edges (α^i, β^i) and the edge (β^i, α^i) are added to G for each i. This represents that β^i equals α^i for all positions to the left of the position under consideration. For each of the other constraints in SC add the variables at the first position as vertices to G. Also add the first position on the left side and the first position on the right side as an edge. Then compute the transitive closure, TC, of G. Check if (α^n, β^n) and (β^n, α^n) belong to TC, if so the position can be removed. If not, (i) select one constraint from SC, $\alpha_1 \leq_{lex} \beta_1$, but not the one under consideration. Let $\alpha_1^k \leq_{lex} \beta_1^k$ be the first position from the left in the constraint where not $\alpha_1^k = \beta_1^k$. Check if $(\alpha_1^k, \beta_1^k) \in TC$. If so, consider next constraint in TC. If (α_1^k, β_1^k) is not in TC, then compute the transitive clousure from $TC_1 = TC \cup \{(\alpha_1^k, \beta_1^k)\}$. Check if both (α^n, β^n) and (β^n, α^n) belongs to TC_1 and if so the position may be removed, otherwise repeat the action described in (i) but with TC_1 instead. When all positions in all constraints has been considered for simplification the process is done.

The algorithm is not very fast, at least not my implementation of it. It did, however, suffice to simplify the constraints from both the $M_{4\times3}$-matrix and the $M_{4\times4}$-matrix, which are the largest matrixes studied in this paper. In the 4×4 case it was necessary to first simplify the constraints individually, due to time and space considerations. The simplified constraints for the 2×3-, 4×3- and 4×4-case are to be found in Table A.1, A.6 and A.12.

2.6 Simplifications — Domain Dependent

2.6.1 Logic Minimization

As we could see in section 2.2 all predicate logical formulas can be expressed in either *conjunctive normal form* (CNF) or in *disjunctive normal form* (DNF). This section briefly describes different approaches in order to minimize such formulas. Experiments in order to find out if the minimized form of the formulas are faster will then be conducted. A definition of minimal form for a formula on both disjunctive and conjunctive normal form is included.

Minimizing a Formula, DNF

In order to understand how to minimize a DNF formula we first need to understand what the size of a DNF formula is. It is common to define the size of a DNF formula as the number of disjuncts that constitutes the formula. Those disjuncts are built up from conjunctions of literals.

Definition 2.6.1. The size of a formula α in DNF is the number of disjuncts that constitutes the formula.

Example Consider the formula $(\alpha_1 \wedge \alpha_2) \vee (\alpha_3 \wedge \alpha_4 \wedge \alpha_5) \vee (\alpha_6 \wedge \alpha_7)$. This formula is in disjunctive normal form and the size of it is 3.

One way to find a DNF representation of a formula α is to first construct its truth table. When this has been constructed it is possible to read a DNF representation from the trut table by checking for which values the formula is true. For each of the true entries form a conjunction of the atomic formulas, and then form a disjunction of those formulas.

Example Consider the truth table of $\alpha_1 \wedge (\alpha_2 \vee \alpha_3)$ shown in table 2.4. This

Table 2.4: Truth table for $\alpha_1 \wedge (\alpha_2 \vee \alpha_3)$

α_1	α_2	α_3	$\alpha_1 \wedge (\alpha_2 \vee \alpha_3)$
true	true	true	true
true	true	false	true
true	false	true	true
true	false	false	false
false	true	true	false
false	true	false	false
false	false	true	false
false	false	false	false

gives the disjunctive normal form $(\alpha_1 \wedge \alpha_2 \wedge \alpha_3) \vee (\alpha_1 \wedge \alpha_2 \wedge \neg\alpha_3) \vee (\alpha_1 \wedge \neg\alpha_2 \wedge \alpha_3)$ of $\alpha_1 \wedge (\alpha_2 \vee \alpha_3)$, with size 3.

In order to get a minimized version of a formula it is possible to use a number of different programs that has been developed for that purpose. One such program is Espresso [14]. The program is in this work considered as a black-box, that is, one gives the program some input and it returns some output, how the output is derived from the input is not considered. It is possible to specify the input to Espresso as a list of the input variables, a list of the output variables, and a truth table for the function to be minimized and the number of lines in the truth table. It is not necessary to specify the complete truth table but only when it exists an output variable which is true. If it then is specified that Espresso should use exact minimization the program returns a minimized formula in disjunctive normal form.

In order to find the minimized disjunctive normal form of the constraints for the $M_{2\times3}$-matrix its truth table, with only the *true* rows included, was first constructed. This truth table was then converted to tt-format, which is the format that Espresso uses. The tt-format includes some information about the names of the input and output variables, and also how many of them there are. For a more complete treatment of the different commands available in Espresso see [14]. In Figure A.3 the minimized DNF of lex-constraints for $M_{2\times3}$, for a domain of size 2 is shown. The different x_i:s in this figure are then used in the SICStus constraint solver for finite domains represented as different equality constraints. For example consider $\neg x_1 \wedge \neg x_3 \wedge x_4 \wedge x_5$, this constraint will be represented as:

```
#\ X1#=1 #/\ #\ X2#=1 #/\ X5#=1
```

where the domain of `X1`, `X2` and `X3` is $\{0, 1\}$ and 0 represents *false* and 1 *true*. Another alternative way to post the constraints to the constraint solver is by replacing the "`#\ X1#=1`" with "`X1#=0`". The representation used in this paper is the first one.

$$
\begin{aligned}
(\neg x_1 \wedge \neg x_3 \wedge x_4 \wedge x_5) \vee \\
(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge x_6) \vee \\
(\neg x_1 \wedge \neg x_3 \wedge x_4 \wedge x_6) \vee \\
(\neg x_1 \wedge x_4 \wedge x_5 \wedge x_6) \vee \\
(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \neg x_4 \wedge \neg x_5) \vee \\
(\neg x_1 \wedge x_2 \wedge x_3 \wedge x_5 \wedge x_6) \vee \\
(x_2 \wedge x_3 \wedge x_4 \wedge x_5 \wedge x_6)
\end{aligned}
$$

Figure 2.4: Minimized DNF of lex-constraints for $M_{2\times 3}$, domain 2

Minimizing a Formula, CNF

As earlier mentioned it is possible to express all predicate logical formulas in either DNF or CNF. In order to find out if a minimized version of CNF is faster than a non-minimized it necessary to know how to get a minimized version of a CNF formula.

The size of a formula in CNF is defined in a similar way as in the DNF case.

Definition 2.6.2. The size of a formula α in CNF is the number of conjuncts that constitutes the formula.

In order to obtain the minimal conjunctive normal form of a formula one can not directly use Espresso. The reason for this is that Espresso only returns the minimal disjunctive normal form of a formula. This can be solved in the following way: consider the problem of obtaining the minimal conjunctive form of the formula α. Then construct the truth table of $\neg\alpha$ and minimize this with Espresso. Let the result of this be β, a minimized disjunctive normal form of $\neg\alpha$. Then use De Morgan's Laws to find $\neg\beta$ in a conjunctive normal form. This form is also a minimal conjunctive form, see Theorem 2.6.3, and equivalent to α, since $\neg(\neg\alpha) \leftrightarrow \alpha$.

Theorem 2.6.3. *Let α be a formula in minimal disjunctive normal form. It is then possible to use De Morgan's Laws in order to find a formula β which is equivalent with $\neg\alpha$ and is in minimal conjunctive normal form.*

Proof. Let α be a formula in minimal disjunctive normal form. It is trivial to apply De Morgan's Law in order to get $\neg\alpha$ in conjunctive normal form. For the sake of contradiction assume that this form is not minimal. It does then exist a shorter form of $\neg\alpha$, let it be β. Then apply De Morgan's Laws on β which will result in $\neg\beta$ in disjunctive minimal form. This formula is equivalent with α and shorter, which contradicts that α is a formula in minimal disjunctive form. □

$$
\begin{array}{r}
(x_1 \wedge \neg x_2) \vee \\
(x_1 \wedge \neg x_3) \vee \\
(x_1 \wedge \neg x_4) \vee \\
(x_3 \wedge \neg x_5) \vee \\
(x_3 \wedge \neg x_6) \vee \\
(\neg x_2 \wedge x_3 \wedge \neg x_4) \vee \\
(x_2 \wedge \neg x_3 \wedge \neg x_4) \vee \\
(\neg x_4 \wedge x_5 \wedge \neg x_6) \vee \\
(x_4 \wedge \neg x_5 \wedge \neg x_6)
\end{array}
$$

Figure 2.5: Minimal DNF of $\neg\alpha$

Let α be the conjunction of the lexicographical constraints in Figure 2.2. In order to find the minimized disjunctive normal form of α, the truth table for $\neg\alpha$ is first constructed. This table is then used with Espresso in order to get a minimal DNF of $\neg\alpha$, see Figure 2.6.1. This formula is then transformed by De Morgan's Laws, for the result see Figure 2.6.1. The problem of obtaining

$$
\begin{array}{r}
(\neg x_1 \vee x_2) \wedge \\
(\neg x_1 \vee x_3) \wedge \\
(\neg x_1 \vee x_4) \wedge \\
(\neg x_3 \vee x_5) \wedge \\
(\neg x_3 \vee x_6) \wedge \\
(x_2 \vee \neg x_3 \vee x_4) \wedge \\
(\neg x_2 \vee x_3 \vee x_4) \wedge \\
(x_4 \vee \neg x_5 \vee x_6) \wedge \\
(\neg x_4 \vee x_5 \vee x_6)
\end{array}
$$

Figure 2.6: Minimal CNF of α

a CNF (DNF) representation from a DNF (CNF) representation is in general NP-complete. The term NP-complete refer to the hardest problems in the class NP. A problem belongs to NP if it exists an algorithm which can guess a solution and then verify if the solution is correct or not in polynomial time, for a more detailed treatment of NP-completeness see [15].

It has been discovered that it is possible to simplify the number of lexicographic constraints necessary for breaking all of the symmetries if the domain of the decision variables in the matrix model is small [9]. The meaning of small in this context is that the domain of the decision variables at least has to be less than the number of decision variables

For example consider $M_{2\times3}$-matrix. If the domain of the decision variable is two it has been shown that it is possible to break all the symmetries in the matrix by using the lexicographic constraints generated from the symmetries of cycle length two [9]. The set of constraints generated from permutations of cycle length two are enough for breaking all the symmetries in this case. Not even all of them are necessary for breaking all the symmetries, but only a subset. In order to see if there is another set of constraints which also breaks the same symmetries, but contains a constraint generated by permutation of a different cycle length than two, consider the following. In order to find a minimal set of lexicographic constraints which break all the symmetries in a matrix it is possible to first generate a set of constraints which is minimal for any domain, let this set be A. Then select a subset $B \subseteq A$, which is of size zero. If this set does not break all the symmetries, which it obviously in this case will not, select a new subset with the size increased by one. If no set of this size break all the symmetries select one with the size increased with one, and so on. Since the subset eventually will include all of the constraints in A it is guaranteed that a minimal set of lexicographic constraints eventually will be found. It has to be added that this method is not efficient, which is an understatement. The reason for this is that the number of sets that will have to be tested in worst case are $|A|^{|A|}$, which for a set of twelve elements gives $12^{12} = 8916100448256$ elements.

Example Consider the matrix in Figure 2.1. A set of which fully break all the symmetries, independent of domain size, are given in Figure 2.3. The two subsets of this set, which breaks all the constraints for domain size two is presented in Table 2.5.

Table 2.5: Completely Simplified *lex*-constraints, $M_{2\times 3}$, domain 2

The First set
$[x_1, x_2, x_3] \leq_{lex} [x_5, x_4, x_6]$
$[x_1, x_4] \leq_{lex} [x_2, x_5]$
$[x_2, x_5] \leq_{lex} [x_3, x_6]$

The Second Set
$[x_1, x_2, x_3, x_4] \leq_{lex} [x_5, x_6, x_4, x_2]$
$[x_1, x_4] \leq_{lex} [x_2, x_5]$
$[x_2, x_5] \leq_{lex} [x_3, x_6]$

An heuristic method for finding a set of constraints which breaks a number of constrints is as follows. Start with a set, A, of constraints, which is guaranteed to remove all of the symmetries for a matrix independent of domain size of the decision variables, and another initially empty set B. Then fix a domain size, two for example. Then remove one constraint, β, from the set A. Check if the new set of constraints still breaks all of the symmetries. If not so add β to B, such a constraint is said to be *non-redundant*. Then remove one element from $A - \beta$ and check if the result in union with B still breaks all of the symmetries, if not add the constraint removed from $A - \beta$ to B. If no constraint can be removed with the result that not all symmetries still remain, increase the number of constraints removed at a time with one. Repeat this process until B contains a set of symmetry breaking predicates which breaks all symmetries for a specific domain size. In practice this process may be considered finished when when for example all subsets up until a fixed size, lets say n, has been considered. The reason for this is because the number of subsets to consider grows quite fast, $\binom{|K|}{i}$, where $|K|$ is the number of elements in the set from which elements are removed and i is the number of elements removed.

Example Consider the matrix in Figure 2.1. A set of which fully break all the symmetries are given in Figure 2.2. Finding all the non-redundant constraints of size one results in the following set.

Table 2.6: Non-redundant, size 1

$[x_1, x_4] \leq_{lex} [x_2, x_5]$
$[x_2, x_5] \leq_{lex} [x_3, x_6]$

The set of constraints in Table 2.6 leaves 20 possible solutions and do not break all the symmetries, since it should have left only 13 distinct solutions in that case. No sets of size two or three breaks any additional symmetries. However, there is four sets of size four which breaks additional symmetries. This leaves two sets of size four which are redundant if one keeps the remaining constraints. The sets are the same as in Table 2.5. This process is unfortunately not practical for finding a minimal set in the 4×3-case, since the number of decision variables are to great.

Chapter 3

Methodology

This chapter will explain the methodology choosen in order to answer the research questions. Litterature studies have been conducted. Relevant articles were provided by my supervisor, head of the ASTRA research group. ASTRA stands for "**A**nalysis, **S**ynthesis, and **T**ransformation / **R**eformulation of **A**lgorithms". Other interesting articles, mostly found by checking the references in the recommended articles has also been used.

The different constraints have been studied in the absence of any constraint satisfaction problem, which would have provided more problem specific constraints. The reason for this is that it is easier to show the impact of the lexicographic constraints by themselves if no other constraints are present. It should, however, be noted that additional constraints might provide a substantial speed-up factor. The clean, constraint satisfaction problems that have been choosen are the one with variables from matrix modells of different sizes. The sizes choosen are 3×2, 4×3 and 4×4. Of those only the 3×2-matrix has earlier been studied to any greater extent. The variables of the matrix have been flattened so that the 3×2-matrix is represented as a list of variables with the same names as in Figure 2.1, on page 16. These variables are then allowed to obtain values from a chosen domain, in most of the experiments of size 2, $\{0, 1\}$. In none of the experiments has any additional constraints been used other than the constraints specified by the particular theory under consideration.

3.1 Used Programs

3.1.1 Prolog, SICStus

SICStus was choosen for the experiments for a number of reasons. It provides constraint solving over finite domains and it also implements lex_chain, which saves a great deal of time because it is not necessary to implement it by hand. Another reason for choosing Prolog is my familiarity and knowledge in the language, which also reduces time needed for development.

3.1.2 GAP – Groups, Algorithms, and Programming

GAP is a system for computional discrete algebra, with particular emphasis on computional group theory. GAP provides a programming language, a huge library of functions implementing algebraic written in the GAP language as well as large data libraies of algebraic objects. The system is distributed freely. In this papper GAP has mainly been used to obtain the lexicographic constraints which later have been considered for simplification. An interface to GAP was provided from start by Justin Pearson, and was after some slight modifications used.

3.1.3 Espresso

Espresso is a program for minimizing locical exprssions. It takes as input a two-level representation of a two-valued (or multiple-valued) Boolean function, and produces a minimal equivalent representation. The minimized version is in DNF and minimized with regard to the number of disjuncts. For further information see [14].

3.2 Hardware

- Processor: Pentium 1400 MHz
- Memory: 256 MB ram
- Operating system: Gentoo, Linux distribution
- Prolog: SICstus, version 3.1.11

Chapter 4

Experiments

In this chapter various results concerning matrices of different sizes are presented. For each of the matrices the result of applying domain independent simplifications and the result of trying to find an approxmative min set as described in section 2.6 is presented. The minimal DNF and CNF form of the constraints is also presented. The different constraints are then compared in a number of different ways.

An explanation to the different tables follows. The domain independent simplifications are presented in one table, with both internal and external simplifications conducted. Those tables are named *Completely Simplified lex-constraints,* $M_{m\times n}$. The complete symmetry groups are excluded because of the number of constraints they include. The $\leq_{lex}$ operator used in the tables is as definied in the definition 2.2.9 on page 15. The domain dependent simplifications include an approximate minimal set of lexicographic constraints for the different sizes of matrices considered. What n has been used in the algorithm for obtaining them is specified in each table. In the domain dependent section the minimal DNF form of the constraints as well as the minimal CNF representation of the constraints are displayed. In this table the symbols specified for and, or and negation are in accordance with the symbols used in section 2.2.

The comparison between the different constraints brings up a number of difficulties. For example is the size of the different constraints a problem. Meauring the size of either the CNF or DNF form of a constraint is fairly straight forward, and uses definition 2.6.2 and 2.6.1. The size of a lexicographic constraint is a little bit more complicated. If one considers each lexicographic constraint as an atomic formula, the result is that the size of the constraint will be the same as the total number of lexicographic constraints included in the set. However, if one first chooses to translate the lexicographic constraints to a logic formula, with the help of definition 2.2.9, not involving special predicates like lex_chain it results in a quadratic representation in the number of variables in the constraint. Even this encoding is unfair since it is possible to use yet another encoding, which is linear in the size of literals of a constraint. This encoding is described in [1], and has a total size of $14n$ literals, where n is the number of different variables in the constraint. It is this encoding that has been used for calculating the size of the lexicograpical constraints because it is at the moment best known encoding of a single constraint.

The different set of constraints are then compared in a number of ways. The tables summarizing theese comparisons are named *Comparision of Constraints, $M_{m\times n}$, domain i* and uses the following:

- *Nr of const* In the CNF- and DNF-case this is in accordance with 2.6.2 and 2.6.1. In the lexicographical constraints it is the number of different lexicograpical constraints.
- *Nr of sol* This is the total amount of distinct matrices found when the constraints are applied.
- *Nr of lit* This is in the CNF and DNF case the number of variables in the respective forms.
- *Speed 1* This is the time in milliseconds to the first solution, measured by taking the avrage of 100 tries. It uses the walltime, which includes memory management and such.
- *Speed 2* This is the time in milliseconds to find all solutions, measured by taking the average of 100 tries. It uses the walltime, which includes memory management and such.

4.1 Conclusions and Future Directions

This section will discuss the different results from the experiments and how the results relate to the different research questions.

- **Is it possible to find a polynomial subset of constraints which breaks most of the symmetries?**
  It is difficult to say anything about this in the general case. The results for domain two seem to indicate that it is possible to break almost all symmetries with relatively few lexicographic constraints, see *the approximated min set* in Table A.5, A.10 and A.16. However, if the used subset of lexicographical constraints is polynomial in size or not is still unanswered. A method for simplifying different lexicographic constraints has been developed and implemented. The method is sufficient for simplifications of matrices up to size 4×4, for larger matrices some further refinement of the method might be needed.
- **Generate and study sets of constraints for larger matrices than the earlier studied largest matrice.**
  Matrices that have been studied are 2×3, 4×3 and 4×4. Data for the different matrices have been included, both data about the different simplifications, both domain independent and domain dependent. Conclusions that might be drawn from this data include that it seems possible to break a large extent of the symmetries by using a relatively small subset of the lexicographic constraints. How large the subset of constraints has to be in order to break most of the symmetries has not been answered and more research is needed for finding the answer of this question. The problem would probably benefit from being attacked from the angle of group theory and then in particular from someone with knowledge in permutation groups.

- **Examine possible domain specific simplifications.**
  The conducted examinations of domain dependent simplifications has resulted in a minimal set of logical constraints for domain size two in the 2×3-, 4×3- and 4×4-case. One problem with this approach is that the input to the program used for minimizing the logical formulas, Espresso, expects its input in a quite extensive format. This may result in problems with larger matrices. A few interesting things has been discovered with the minimal set of logical constraints. In the 2×3-case and the 4×3-case it has been shown that the set of lexicographic constraints with all domain independendent simplifications carried out results in a faster average time for finding all solutions. In the 4×4-case the opposite is true, and the minimized set of logical formulas are about twice as fast. (When discussing the minimized set of logical formulas, it is the set in CNF and not in DNF that is considered. The CNF form has been shown to be faster for finding all solutions, independent of the size of the matrix, see Table A.5, A.10 and A.16.) The interesting part is that the minimized set of logical formulas not are faster even for the smaller matrices, the reason why one could expect that so should be the case is that those set are fully simplified with consideration taken to that the domain is of size two.

  For the matrix sizes where a minimal set of lexicograhic constraints, which breaks all of the symmetries, has been found it has been shown that the found set is faster than the minimized set of logical formulas in CNF form, at least twice as fast in average for finding all the solutions and average time to first solution, see Table A.5. One possible way to increase the speed for the minimized set of logical formulas could be to encode `\# X#=1` as `X#=0`, but no experiments to decide what effects this would have on the speed have been conducted. In the cases where no minimal set of lexicographic constraints has been found and instead an approximative set has been constructed, it has been shown that this set is substantially faster than the corresponding set of logical formulas in CNF. In the 4×3-case about three times as fast and in the 4×4-case about five times as fast for finding all the solutions in average.

  Comparisons between the approximate minimal set of lexicographic constraints and the commonly used lex^2-constraint shows that they are approximately even in the 2×3-case, and that lex^2 is somewhat faster in the 4×3-case, with domain two, in finding all solutions in average, and substantially faster in finding the first solution. This is not suprising since there are fewer constraints to satisfy in the lex^2-case and hence fewer constraints for the constraint solver to satisfy. When the domain is of size three the approximative minimal set is somewhat faster, the reason for this is that this set breaks much more of the symmetries compared to the set of lex^2-constraints, see Table A.11. The gap between the approximate minimal set and the lex^2-set of constraints for finding all solutions will probably increase even more with a larger domain size.

4.2 Original Work

In this section the orginal results for this papper is considered

- A rule which can be used for simplifying lexicographical expressions has been deviced. This rule supercedes the earlier rules deviced in [11]. Proof that this rule is correct and strictly stronger is also given.
- An algorithm which mechanize the simplifications of lexicographical constraints has been constructed. Such an agortihm has not earlier existed.
- Logical simplifications of $M_{4\times3}$ and $M_{4\times4}$ matrices have been conducted, resulting in a minimal set of lexicographical constraints for thoose matrices. Lexicographical constraints for thoose matrices has not earlier been fully simplified.
- A method for breaking symmetries in matrice models (with the domain sixe 2) by minimized logical formulas in DNF and CNF form has been examined and compared with breaking the same symmetries with lexicographic constraints. This has not earlier been done.
- A method for finding an approximated minimal set of lexicographical constraints, for a specific domain size, which experimentaly has been shown to break most of the symmetries for the considered matrices has been deviced.

Appendix A

Tables for Different Matrices

A.1 2×3-Matrix Models

Table A.1: Completely Simplified *lex*-constraints, $M_{2\times3}$

$[x_1, x_2, x_3] \leq_{lex} [x_4, x_5, x_6]$
$[x_1, x_2, x_3] \leq_{lex} [x_6, x_5, x_4]$
$[x_1, x_2, x_3] \leq_{lex} [x_6, x_4, x_5]$
$[x_1, x_2, x_3] \leq_{lex} [x_5, x_4, x_6]$
$[x_1, x_2, x_3, x_4] \leq_{lex} [x_5, x_6, x_4, x_2]$
$[x_1, x_2, x_3] \leq_{lex} [x_4, x_6, x_5]$
$[x_1, x_4] \leq_{lex} [x_2, x_5]$
$[x_2, x_5] \leq_{lex} [x_3, x_6]$

Table A.2: Completely Simplified *lex*-constraints, $M_{2\times3}$, domain 2

The First set
$[x_1, x_2, x_3] \leq_{lex} [x_5, x_4, x_6]$
$[x_1, x_4] \leq_{lex} [x_2, x_5]$
$[x_2, x_5] \leq_{lex} [x_3, x_6]$

The Second Set
$[x_1, x_2, x_3, x_4] \leq_{lex} [x_5, x_6, x_4, x_2]$
$[x_1, x_4] \leq_{lex} [x_2, x_5]$
$[x_2, x_5] \leq_{lex} [x_3, x_6]$

Table A.3: Minimized DNF of lex-constraints for $M_{2\times3}$, domain 2

$$
\begin{aligned}
(\neg x_1 \wedge \neg x_3 \wedge x_4 \wedge x_5) \vee \\
(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge x_6) \vee \\
(\neg x_1 \wedge \neg x_3 \wedge x_4 \wedge x_6) \vee \\
(\neg x_1 \wedge x_4 \wedge x_5 \wedge x_6) \vee \\
(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \neg x_4 \wedge \neg x_5) \vee \\
(\neg x_1 \wedge x_2 \wedge x_3 \wedge x_5 \wedge x_6) \vee \\
(x_2 \wedge x_3 \wedge x_4 \wedge x_5 \wedge x_6)
\end{aligned}
$$

Table A.4: Minimized CNF of lex-constraints for $M_{2\times3}$, domain 2

$$
\begin{aligned}
(\neg x_1 \vee x_2) \wedge \\
(\neg x_1 \vee x_3) \wedge \\
(\neg x_1 \vee x_4) \wedge \\
(\neg x_3 \vee x_5) \wedge \\
(\neg x_3 \vee x_6) \wedge \\
(x_2 \vee \neg x_3 \vee x_4) \wedge \\
(\neg x_2 \vee x_3 \vee x_4) \wedge \\
(x_4 \vee \neg x_5 \vee x_6) \wedge \\
(\neg x_4 \vee x_5 \vee x_6)
\end{aligned}
$$

Table A.5: Comparision of Constraints, $M_{2\times3}$, domain 2

	Nr of const	Nr of sol	Nr of lit	Speed	Speed
Espresso - exact CNF	9	13	22	4.10	4.80
Espresso - exact DNF	7	13	31	5.40	9.60
Lex, entire symmetry group	12	13	1008	4.30	5.20
Lex, after simplifications	8	13	322	2.20	2.90
Lex, approximated min set, 1	3	13	98	1.30	1.90
Lex, approximated min set, 2	3	13	112	1.30	2.00
Lex, row-col	3	14	98	1.2	1.90

A.2 4x3-Matrix Models

Table A.6: Completely Simplified *lex*-constraints, $M_{4\times 3}$

$[x_1, x_2, x_3, x_4, x_7]$	$\leq_{lex}$	$[x_{12}, x_{11}, x_{10}, x_6, x_9]$
$[x_1, x_4, x_5, x_6, x_{10}]$	$\leq_{lex}$	$[x_3, x_9, x_8, x_7, x_{12}]$
$[x_1, x_2, x_3, x_4, x_7]$	$\leq_{lex}$	$[x_{11}, x_{10}, x_{12}, x_5, x_8]$
$[x_1, x_4, x_5, x_6, x_{10}]$	$\leq_{lex}$	$[x_2, x_8, x_7, x_9, x_{11}]$
$[x_1, x_2, x_3, x_5, x_8]$	$\leq_{lex}$	$[x_{10}, x_{12}, x_{11}, x_6, x_9]$
$[x_1, x_2, x_3, x_4, x_{10}]$	$\leq_{lex}$	$[x_9, x_8, x_7, x_6, x_{12}]$
$[x_1, x_4, x_5, x_6, x_7]$	$\leq_{lex}$	$[x_3, x_{12}, x_{11}, x_{10}, x_9]$
$[x_1, x_2, x_3, x_4, x_{10}]$	$\leq_{lex}$	$[x_8, x_7, x_9, x_5, x_{11}]$
$[x_1, x_4, x_5, x_6, x_7]$	$\leq_{lex}$	$[x_2, x_{11}, x_{10}, x_{12}, x_8]$
$[x_1, x_2, x_3, x_5, x_{11}]$	$\leq_{lex}$	$[x_7, x_9, x_8, x_6, x_{12}]$
$[x_2, x_4, x_5, x_6, x_8]$	$\leq_{lex}$	$[x_3, x_{10}, x_{12}, x_{11}, x_9]$
$[x_1, x_2, x_3, x_4, x_5, x_6]$	$\leq_{lex}$	$[x_9, x_8, x_7, x_{12}, x_{11}, x_{10}]$
$[x_1, x_2, x_3, x_7, x_{10}]$	$\leq_{lex}$	$[x_6, x_5, x_4, x_9, x_{12}]$
$[x_1, x_2, x_3, x_4, x_5, x_6]$	$\leq_{lex}$	$[x_{12}, x_{11}, x_{10}, x_9, x_8, x_7]$
$[x_1, x_4, x_7, x_8, x_9]$	$\leq_{lex}$	$[x_3, x_6, x_{12}, x_{11}, x_{10}]$
$[x_1, x_2, x_3, x_4, x_5, x_6]$	$\leq_{lex}$	$[x_8, x_7, x_9, x_{11}, x_{10}, x_{12}]$
$[x_1, x_2, x_3, x_7, x_{10}]$	$\leq_{lex}$	$[x_5, x_4, x_6, x_8, x_{11}]$
$[x_1, x_2, x_3, x_4, x_5, x_6]$	$\leq_{lex}$	$[x_{11}, x_{10}, x_{12}, x_8, x_7, x_9]$
$[x_1, x_4, x_7, x_8, x_9]$	$\leq_{lex}$	$[x_2, x_5, x_{11}, x_{10}, x_{12}]$
$[x_1, x_2, x_3, x_4, x_5, x_6]$	$\leq_{lex}$	$[x_7, x_9, x_8, x_{10}, x_{12}, x_{11}]$
$[x_1, x_2, x_3, x_5, x_6]$	$\leq_{lex}$	$[x_{10}, x_{12}, x_{11}, x_9, x_8]$
$[x_2, x_5, x_7, x_8, x_9]$	$\leq_{lex}$	$[x_3, x_6, x_{10}, x_{12}, x_{11}]$
$[x_1, x_2, x_3, x_7, x_8, x_9]$	$\leq_{lex}$	$[x_6, x_5, x_4, x_{12}, x_{11}, x_{10}]$
$[x_1, x_2, x_3, x_7, x_8, x_9]$	$\leq_{lex}$	$[x_5, x_4, x_6, x_{11}, x_{10}, x_{12}]$
$[x_1, x_2, x_3, x_7, x_8, x_9]$	$\leq_{lex}$	$[x_4, x_6, x_5, x_{10}, x_{12}, x_{11}]$
$[x_2, x_5, x_8, x_{11}]$	$\leq_{lex}$	$[x_3, x_6, x_9, x_{12}]$
$[x_1, x_2, x_3, x_4, x_5, x_7, x_8]$	$\leq_{lex}$	$[x_{12}, x_{10}, x_{11}, x_6, x_4, x_9, x_7]$
$[x_1, x_2, x_3, x_4, x_5, x_6]$	$\leq_{lex}$	$[x_8, x_9, x_7, x_2, x_3, x_1]$
$[x_1, x_2, x_3, x_4, x_5, x_7, x_8, x_{10}]$	$\leq_{lex}$	$[x_{11}, x_{12}, x_{10}, x_5, x_6, x_8, x_9, x_2]$
$[x_1, x_2, x_4, x_5, x_6, x_7]$	$\leq_{lex}$	$[x_2, x_3, x_8, x_9, x_7, x_5]$
$[x_1, x_2, x_3]$	$\leq_{lex}$	$[x_4, x_5, x_6]$
$[x_4, x_5, x_6]$	$\leq_{lex}$	$[x_7, x_8, x_9]$
$[x_1, x_4, x_5, x_6, x_7, x_8]$	$\leq_{lex}$	$[x_3, x_{12}, x_{10}, x_{11}, x_9, x_7]$
$[x_1, x_2, x_3, x_4, x_5, x_7]$	$\leq_{lex}$	$[x_8, x_9, x_7, x_5, x_6, x_2]$
$[x_1, x_2, x_3, x_4, x_5, x_6]$	$\leq_{lex}$	$[x_{11}, x_{12}, x_{10}, x_2, x_3, x_1]$
$[x_1, x_2, x_4, x_5, x_6, x_7, x_8, x_{10}]$	$\leq_{lex}$	$[x_2, x_3, x_{11}, x_{12}, x_{10}, x_8, x_9, x_5]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8]$	$\leq_{lex}$	$[x_9, x_7, x_8, x_{12}, x_{10}, x_{11}, x_6, x_4]$
$[x_1, x_4, x_7, x_8, x_9]$	$\leq_{lex}$	$[x_3, x_6, x_{12}, x_{10}, x_{11}]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7]$	$\leq_{lex}$	$[x_8, x_9, x_7, x_{11}, x_{12}, x_{10}, x_5]$
$[x_1, x_2, x_3, x_4]$	$\leq_{lex}$	$[x_5, x_6, x_4, x_2]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7]$	$\leq_{lex}$	$[x_{11}, x_{12}, x_{10}, x_8, x_9, x_7, x_2]$
$[x_1, x_2, x_4, x_5, x_7, x_8, x_9, x_{10}]$	$\leq_{lex}$	$[x_2, x_3, x_5, x_6, x_{11}, x_{12}, x_{10}, x_8]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8]$	$\leq_{lex}$	$[x_6, x_4, x_5, x_{12}, x_{10}, x_{11}, x_9, x_7]$
$[x_1, x_4, x_5, x_6, x_7, x_8]$	$\leq_{lex}$	$[x_3, x_9, x_7, x_8, x_{12}, x_{10}]$
$[x_1, x_2, x_3, x_4, x_5, x_{10}]$	$\leq_{lex}$	$[x_8, x_9, x_7, x_2, x_3, x_{11}]$

Table A.6 – Continued on next page

Table A.6 – continued from previous page

$[x_1, x_2, x_3, x_4, x_5, x_7, x_8, x_9]$	$\leq_{lex}$	$[x_{11}, x_{12}, x_{10}, x_5, x_6, x_2, x_3, x_1]$
$[x_1, x_2, x_3, x_4, x_5, x_7, x_8]$	$\leq_{lex}$	$[x_9, x_7, x_8, x_6, x_4, x_{12}, x_{10}]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_{10}, x_{11}]$	$\leq_{lex}$	$[x_6, x_4, x_5, x_9, x_7, x_8, x_{12}, x_{10}]$
$[x_1, x_2, x_3, x_4, x_5, x_7]$	$\leq_{lex}$	$[x_{11}, x_{12}, x_{10}, x_2, x_3, x_8]$
$[x_1, x_2, x_4, x_5, x_6, x_7, x_8, x_9]$	$\leq_{lex}$	$[x_2, x_3, x_{11}, x_{12}, x_{10}, x_5, x_6, x_4]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8]$	$\leq_{lex}$	$[x_9, x_7, x_8, x_{12}, x_{10}, x_{11}, x_3, x_1]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8]$	$\leq_{lex}$	$[x_{12}, x_{10}, x_{11}, x_9, x_7, x_8, x_6, x_4]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7]$	$\leq_{lex}$	$[x_8, x_9, x_7, x_{11}, x_{12}, x_{10}, x_2]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7]$	$\leq_{lex}$	$[x_{11}, x_{12}, x_{10}, x_8, x_9, x_7, x_5]$
$[x_2, x_4, x_5, x_6, x_{11}]$	$\leq_{lex}$	$[x_3, x_7, x_9, x_8, x_{12}]$
$[x_1, x_2, x_3, x_8, x_{11}]$	$\leq_{lex}$	$[x_4, x_6, x_5, x_9, x_{12}]$
$[x_1, x_4, x_7, x_{10}]$	$\leq_{lex}$	$[x_2, x_5, x_8, x_{11}]$
$[x_7, x_8, x_9]$	$\leq_{lex}$	$[x_{10}, x_{11}, x_{12}]$

Table A.7: Approximate minimal set of *lex*-constraints, $M_{4\times3}$, domain 2

$[x_7, x_8, x_9]$	$\leq_{lex}$	$[x_{10}, x_{11}, x_{12}]$
$[x_1, x_4, x_7, x_{10}]$	$\leq_{lex}$	$[x_2, x_5, x_8, x_{11}]$
$[x_1, x_2, x_3, x_8, x_{11}]$	$\leq_{lex}$	$[x_4, x_6, x_5, x_9, x_{12}]$
$[x_2, x_4, x_5, x_6, x_{11}]$	$\leq_{lex}$	$[x_3, x_7, x_9, x_8, x_{12}]$
$[x_4, x_5, x_6]$	$\leq_{lex}$	$[x_7, x_8, x_9]$
$[x_2, x_5, x_8, x_{11}]$	$\leq_{lex}$	$[x_3, x_6, x_9, x_{12}]$
$[x_1, x_4, x_7, x_8, x_9]$	$\leq_{lex}$	$[x_2, x_5, x_{11}, x_{10}, x_{12}]$
$[x_1, x_2, x_4, x_5, x_6, x_7, x_8, x_9]$	$\leq_{lex}$	$[x_2, x_3, x_{11}, x_{12}, x_{10}, x_5, x_6, x_4]$
$[x_1, x_4, x_5, x_6, x_{10}]$	$\leq_{lex}$	$[x_2, x_8, x_7, x_9, x_{11}]$
$[x_1, x_2, x_3, x_4, x_5, x_{10}]$	$\leq_{lex}$	$[x_8, x_9, x_7, x_2, x_3, x_{11}]$
$[x_1, x_2, x_3, x_4, x_5, x_6]$	$\leq_{lex}$	$[x_{11}, x_{12}, x_{10}, x_2, x_3, x_1]$
$[x_1, x_2, x_3, x_4, x_5, x_6]$	$\leq_{lex}$	$[x_8, x_9, x_7, x_2, x_3, x_1]$
$[x_1, x_2, x_3, x_7, x_8, x_9]$	$\leq_{lex}$	$[x_5, x_4, x_6, x_{11}, x_{10}, x_{12}]$
$[x_1, x_4, x_5, x_6, x_7]$	$\leq_{lex}$	$[x_2, x_{11}, x_{10}, x_{12}, x_8]$

Table A.8: Minimized DNF of lex-constraints for $M_{4\times3}$, domain 2

$$
\begin{aligned}
&(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \neg x_4 \wedge \neg x_5 \wedge \neg x_7 \wedge \neg x_8 \wedge x_9 \wedge x_{11}) \vee \\
&(\neg x_1 \wedge \neg x_2 \wedge \neg x_4 \wedge \neg x_5 \wedge x_6 \wedge \neg x_7 \wedge \neg x_8 \wedge x_9 \wedge x_{11}) \vee \\
&(\neg x_1 \wedge \neg x_2 \wedge \neg x_4 \wedge \neg x_5 \wedge x_6 \wedge \neg x_7 \wedge x_9 \wedge x_{10} \wedge x_{11}) \vee \\
&(\neg x_1 \wedge \neg x_2 \wedge \neg x_4 \wedge x_6 \wedge \neg x_7 \wedge x_8 \wedge x_9 \wedge x_{10} \wedge x_{12}) \vee \\
&(\neg x_1 \wedge \neg x_2 \wedge \neg x_4 \wedge \neg x_5 \wedge x_6 \wedge \neg x_7 \wedge x_8 \wedge x_{11} \wedge x_{12}) \vee \\
&(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \neg x_4 \wedge \neg x_5 \wedge \neg x_7 \wedge x_9 \wedge x_{11} \wedge x_{12}) \vee \\
&(\neg x_1 \wedge \neg x_2 \wedge \neg x_4 \wedge x_6 \wedge \neg x_7 \wedge x_8 \wedge x_9 \wedge x_{11} \wedge x_{12}) \vee \\
&(\neg x_1 \wedge \neg x_2 \wedge x_3 \wedge \neg x_4 \wedge \neg x_5 \wedge x_6 \wedge \neg x_7 \wedge x_8 \wedge \neg x_9 \wedge x_{11}) \vee \\
&(\neg x_1 \wedge \neg x_2 \wedge x_3 \wedge \neg x_4 \wedge x_6 \wedge \neg x_7 \wedge x_8 \wedge x_9 \wedge x_{10} \wedge x_{11}) \vee \\
&(\neg x_1 \wedge \neg x_2 \wedge x_3 \wedge \neg x_4 \wedge x_5 \wedge x_7 \wedge \neg x_8 \wedge x_9 \wedge x_{10} \wedge x_{11}) \vee \\
&(\neg x_1 \wedge \neg x_2 \wedge \neg x_4 \wedge x_5 \wedge x_6 \wedge x_7 \wedge \neg x_8 \wedge x_9 \wedge x_{10} \wedge x_{11}) \vee \\
&(\neg x_1 \wedge x_3 \wedge \neg x_4 \wedge x_5 \wedge x_6 \wedge x_7 \wedge \neg x_8 \wedge x_9 \wedge x_{10} \wedge x_{11}) \vee \\
&(\neg x_1 \wedge \neg x_2 \wedge x_3 \wedge \neg x_4 \wedge x_6 \wedge x_7 \wedge x_8 \wedge \neg x_9 \wedge x_{10} \wedge x_{11}) \vee \\
&(\neg x_1 \wedge \neg x_2 \wedge \neg x_4 \wedge \neg x_5 \wedge x_6 \wedge x_7 \wedge x_8 \wedge \neg x_9 \wedge x_{10} \wedge x_{11}) \vee \\
&(\neg x_1 \wedge \neg x_2 \wedge \neg x_4 \wedge \neg x_5 \wedge x_6 \wedge \neg x_7 \wedge x_8 \wedge \neg x_9 \wedge x_{10} \wedge \neg x_{11}) \vee \\
&(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \neg x_4 \wedge \neg x_5 \wedge \neg x_7 \wedge x_8 \wedge x_9 \wedge x_{10} \wedge x_{12}) \vee \\
&(\neg x_1 \wedge x_3 \wedge \neg x_4 \wedge x_5 \wedge x_6 \wedge \neg x_7 \wedge x_8 \wedge x_9 \wedge x_{10} \wedge x_{12}) \vee \\
&(\neg x_1 \wedge \neg x_2 \wedge \neg x_4 \wedge \neg x_5 \wedge x_6 \wedge \neg x_7 \wedge \neg x_8 \wedge x_9 \wedge \neg x_{10} \wedge x_{12}) \vee \\
&(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \neg x_4 \wedge \neg x_5 \wedge \neg x_6 \wedge \neg x_7 \wedge \neg x_8 \wedge x_{11} \wedge x_{12}) \vee \\
&(\neg x_1 \wedge \neg x_2 \wedge x_3 \wedge \neg x_4 \wedge x_5 \wedge \neg x_7 \wedge x_8 \wedge x_9 \wedge x_{11} \wedge x_{12}) \vee \\
&(\neg x_1 \wedge x_3 \wedge \neg x_4 \wedge x_5 \wedge x_6 \wedge \neg x_7 \wedge x_8 \wedge x_9 \wedge x_{11} \wedge x_{12}) \vee \\
&(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \neg x_4 \wedge \neg x_5 \wedge x_8 \wedge x_9 \wedge x_{10} \wedge x_{11} \wedge x_{12}) \vee \\
&(\neg x_1 \wedge \neg x_2 \wedge x_5 \wedge x_6 \wedge x_7 \wedge x_8 \wedge x_9 \wedge x_{10} \wedge x_{11} \wedge x_{12}) \vee \\
&(\neg x_1 \wedge \neg x_2 \wedge x_3 \wedge x_4 \wedge x_5 \wedge \neg x_6 \wedge x_7 \wedge x_8 \wedge \neg x_9 \wedge x_{10} \wedge x_{11}) \vee \\
&(\neg x_1 \wedge \neg x_2 \wedge x_3 \wedge \neg x_4 \wedge x_5 \wedge \neg x_6 \wedge \neg x_7 \wedge x_8 \wedge x_9 \wedge x_{10} \wedge \neg x_{11}) \vee \\
&(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \neg x_4 \wedge \neg x_5 \wedge \neg x_6 \wedge \neg x_7 \wedge \neg x_8 \wedge \neg x_9 \wedge \neg x_{10} \wedge \neg x_{11}) \vee \\
&(\neg x_1 \wedge x_2 \wedge x_3 \wedge \neg x_4 \wedge x_5 \wedge x_6 \wedge x_7 \wedge \neg x_8 \wedge x_9 \wedge x_{10} \wedge x_{12}) \vee \\
&(\neg x_1 \wedge \neg x_2 \wedge x_3 \wedge \neg x_4 \wedge x_5 \wedge \neg x_6 \wedge x_7 \wedge \neg x_8 \wedge x_9 \wedge x_{10} \wedge x_{12}) \vee \\
&(\neg x_1 \wedge x_2 \wedge x_3 \wedge x_4 \wedge \neg x_5 \wedge x_6 \wedge x_7 \wedge x_8 \wedge x_{10} \wedge x_{11} \wedge x_{12}) \vee \\
&(\neg x_1 \wedge \neg x_2 \wedge x_3 \wedge \neg x_4 \wedge x_5 \wedge \neg x_6 \wedge x_7 \wedge \neg x_8 \wedge x_{10} \wedge x_{11} \wedge x_{12}) \vee \\
&(x_2 \wedge x_3 \wedge x_4 \wedge x_5 \wedge x_6 \wedge x_7 \wedge x_8 \wedge x_9 \wedge x_{10} \wedge x_{11} \wedge x_{12}) \vee \\
&(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \neg x_4 \wedge \neg x_5 \wedge \neg x_7 \wedge \neg x_8 \wedge x_9 \wedge \neg x_{10} \wedge x_{12}) \vee \\
&(\neg x_1 \wedge \neg x_2 \wedge x_3 \wedge \neg x_4 \wedge x_6 \wedge x_7 \wedge x_8 \wedge x_{10} \wedge x_{11} \wedge x_{12}) \vee \\
&(\neg x_1 \wedge \neg x_2 \wedge x_3 \wedge x_5 \wedge x_7 \wedge x_8 \wedge x_9 \wedge x_{10} \wedge x_{11} \wedge x_{12}) \vee \\
&(\neg x_1 \wedge x_3 \wedge x_5 \wedge x_6 \wedge x_7 \wedge x_8 \wedge x_9 \wedge x_{10} \wedge x_{11} \wedge x_{12})
\end{aligned}
$$

Table A.9: Minimized CNF of lex-constraints for $M_{4\times 3}$, domain 2

$$
\begin{aligned}
&(\neg x_2 \vee \neg x_5 \vee x_9) \wedge \\
&(\neg x_7 \vee x_9 \vee x_{11}) \wedge \\
&(x_3 \vee \neg x_5 \vee \neg x_8 \vee x_{12}) \wedge \\
&(\neg x_4 \vee \neg x_5 \vee \neg x_6 \vee x_9) \wedge \\
&(x_8 \vee \neg x_{10} \vee x_{11} \vee x_{12}) \wedge \\
&(x_5 \vee \neg x_9 \vee x_{11} \vee x_{12}) \wedge \\
&(x_2 \vee \neg x_6 \vee \neg x_7 \vee x_{11}) \wedge \\
&(x_6 \vee \neg x_8 \vee \neg x_9 \vee \neg x_{11} \vee x_{12}) \wedge \\
&(x_7 \vee x_8 \vee \neg x_{10} \vee x_{11}) \wedge \\
&(\neg x_2 \vee \neg x_8 \vee x_{12}) \wedge \\
&(x_8 \vee x_9 \vee \neg x_{10} \vee x_{12}) \wedge \\
&(\neg x_6 \vee \neg x_9 \vee x_{11} \vee x_{12}) \wedge \\
&(\neg x_3 \vee x_5 \vee x_6) \wedge \\
&(\neg x_5 \vee x_7 \vee x_8) \wedge \\
&(x_5 \vee \neg x_7 \vee x_8) \wedge \\
&(\neg x_5 \vee x_7 \vee x_9) \wedge \\
&(\neg x_7 \vee \neg x_8 \vee \neg x_9 \vee x_{12}) \wedge \\
&(x_3 \vee \neg x_5 \vee x_6) \wedge \\
&(\neg x_7 \vee x_{10}) \wedge \\
&(\neg x_6 \vee x_8 \vee x_9) \wedge \\
&(\neg x_4 \vee x_7) \wedge \\
&(\neg x_7 \vee \neg x_8 \vee x_{11}) \wedge \\
&(\neg x_8 \vee \neg x_9 \vee x_{10} \vee x_{12}) \wedge \\
&(\neg x_1 \vee x_4) \wedge \\
&(x_3 \vee \neg x_5 \vee x_9) \wedge \\
&(\neg x_2 \vee x_4 \vee x_5) \wedge \\
&(x_2 \vee \neg x_4 \vee x_5) \wedge \\
&(\neg x_2 \vee x_3) \wedge \\
&(\neg x_8 \vee x_{10} \vee x_{11}) \wedge \\
&(x_3 \vee x_7 \vee x_9 \vee \neg x_{11} \vee x_{12}) \wedge \\
&(x_4 \vee x_6 \vee \neg x_8 \vee x_9) \wedge \\
&(\neg x_2 \vee x_6) \wedge \\
&(\neg x_4 \vee x_8) \wedge \\
&(\neg x_1 \vee x_2) \wedge \\
&(\neg x_1 \vee x_5)
\end{aligned}
$$

Table A.10: Comparision of Constraints, $M_{4\times3}$, Domain 2

	Nr of const	Nr of sol	Nr of lit	Speed	Speed
Espresso - exact CNF	35	87	112	17.80	31.00
Espresso - exact DNF	35	87	363	28.90	84.80
Lex, entire symmetry group	144	87	24192	88.60	124.20
Lex, after simplifications	58	87	4746	18.20	29.20
Lex, approximated min set[a]	14	89	994	4.60	10.40
Lex, row-col	5	130	238	1.80	7.70

[a]The largest n used in the approximated minimal set is 2

Table A.11: Comparision of Constraints, $M_{4\times3}$, domain 3

	Nr of const	Nr of sol	Nr of lit	Speed	Speed
Lex, entire symmetry group	144	5053	24192	88.70	650.90
Lex, after simplifications	58	5053	4746	18.10	291.90
Lex, approximated min set[a]	14	5719	994	4.70	223.10
Lex, row-col	5	10020	238	1.80	333,30

[a]The largest n used in the approximated minimal set is 2

A.3 4x4-Matrix Models

Table A.12: Completely Simplified *lex*-constraints

$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}]$	$\leq_{lex}$	$[x_7, x_5, x_8, x_6, x_{15}, x_{13}, x_{16}, x_{14}, x_3, x_1, x_4, x_2]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_9, x_{10}, x_{11}]$	$\leq_{lex}$	$[x_{15}, x_{13}, x_{16}, x_{14}, x_7, x_5, x_8, x_{11}, x_9, x_{12}]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}]$	$\leq_{lex}$	$[x_{10}, x_{12}, x_9, x_{11}, x_2, x_4, x_1, x_3, x_{14}, x_{16}, x_{13}]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7]$	$\leq_{lex}$	$[x_6, x_8, x_5, x_7, x_{14}, x_{16}, x_{13}]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_9, x_{10}, x_{11}, x_{13}]$	$\leq_{lex}$	$[x_{14}, x_{16}, x_{13}, x_{15}, x_6, x_8, x_5, x_{10}, x_{12}, x_9, x_2]$
$[x_1, x_2, x_5, x_6, x_7, x_8, x_9]$	$\leq_{lex}$	$[x_2, x_4, x_{10}, x_{12}, x_9, x_{11}, x_6]$
$[x_1, x_2, x_3, x_4, x_5, x_9]$	$\leq_{lex}$	$[x_{16}, x_{14}, x_{15}, x_{13}, x_8, x_{12}]$
$[x_1, x_5, x_6, x_7, x_8, x_{13}]$	$\leq_{lex}$	$[x_4, x_{12}, x_{10}, x_{11}, x_9, x_{16}]$
$[x_1, x_2, x_3, x_4, x_6, x_{10}]$	$\leq_{lex}$	$[x_{13}, x_{15}, x_{14}, x_{16}, x_7, x_{11}]$
$[x_2, x_5, x_6, x_7, x_8, x_{14}]$	$\leq_{lex}$	$[x_3, x_9, x_{11}, x_{10}, x_{12}, x_{15}]$
$[x_1, x_3, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}]$	$\leq_{lex}$	$[x_3, x_4, x_{15}, x_{13}, x_{16}, x_{14}, x_{11}, x_9, x_{12}]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_9]$	$\leq_{lex}$	$[x_{10}, x_{12}, x_9, x_{11}, x_6, x_8, x_5, x_2]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8]$	$\leq_{lex}$	$[x_{14}, x_{16}, x_{13}, x_{15}, x_2, x_4, x_1, x_3]$
$[x_1, x_2, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{13}]$	$\leq_{lex}$	$[x_2, x_4, x_{14}, x_{16}, x_{13}, x_{15}, x_{10}, x_{12}, x_9, x_6]$
$[x_1, x_2, x_3, x_4, x_5, x_{13}]$	$\leq_{lex}$	$[x_{12}, x_{10}, x_{11}, x_9, x_8, x_{16}]$
$[x_1, x_5, x_6, x_7, x_8, x_9]$	$\leq_{lex}$	$[x_4, x_{16}, x_{14}, x_{15}, x_{13}, x_{12}]$
$[x_1, x_2, x_3, x_4, x_6, x_{14}]$	$\leq_{lex}$	$[x_9, x_{11}, x_{10}, x_{12}, x_7, x_{15}]$
$[x_2, x_5, x_6, x_7, x_8, x_{10}]$	$\leq_{lex}$	$[x_3, x_{13}, x_{15}, x_{14}, x_{16}, x_{11}]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}]$	$\leq_{lex}$	$[x_{11}, x_9, x_{12}, x_{10}, x_{15}, x_{13}, x_{16}, x_{14}, x_7, x_5]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}]$	$\leq_{lex}$	$[x_{15}, x_{13}, x_{16}, x_{14}, x_{11}, x_9, x_{12}, x_{10}, x_3, x_1]$
$[x_1, x_3, x_5, x_7, x_9, x_{10}, x_{11}, x_{12}]$	$\leq_{lex}$	$[x_3, x_4, x_7, x_8, x_{15}, x_{13}, x_{16}, x_{14}]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9]$	$\leq_{lex}$	$[x_{10}, x_{12}, x_9, x_{11}, x_{14}, x_{16}, x_{13}, x_{15}, x_6]$
$[x_1, x_2, x_3, x_4, x_5]$	$\leq_{lex}$	$[x_6, x_8, x_5, x_7, x_2]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9]$	$\leq_{lex}$	$[x_{14}, x_{16}, x_{13}, x_{15}, x_{10}, x_{12}, x_9, x_{11}, x_2]$
$[x_1, x_2, x_5, x_6, x_9, x_{10}, x_{11}, x_{12}, x_{13}]$	$\leq_{lex}$	$[x_2, x_4, x_6, x_8, x_{14}, x_{16}, x_{13}, x_{15}, x_{10}]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8]$	$\leq_{lex}$	$[x_{12}, x_{10}, x_{11}, x_9, x_{16}, x_{14}, x_{15}, x_{13}]$
$[x_1, x_2, x_3, x_4, x_9, x_{13}]$	$\leq_{lex}$	$[x_8, x_6, x_7, x_5, x_{12}, x_{16}]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8]$	$\leq_{lex}$	$[x_{16}, x_{14}, x_{15}, x_{13}, x_{12}, x_{10}, x_{11}, x_9]$
$[x_1, x_5, x_9, x_{10}, x_{11}, x_{12}]$	$\leq_{lex}$	$[x_4, x_8, x_{16}, x_{14}, x_{15}, x_{13}]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8]$	$\leq_{lex}$	$[x_9, x_{11}, x_{10}, x_{12}, x_{13}, x_{15}, x_{14}, x_{16}]$
$[x_1, x_2, x_3, x_4, x_{10}, x_{14}]$	$\leq_{lex}$	$[x_5, x_7, x_6, x_8, x_{11}, x_{15}]$
$[x_1, x_2, x_3, x_4, x_6, x_7, x_8]$	$\leq_{lex}$	$[x_{13}, x_{15}, x_{14}, x_{16}, x_{11}, x_{10}, x_{12}]$
$[x_2, x_6, x_9, x_{10}, x_{11}, x_{12}]$	$\leq_{lex}$	$[x_3, x_7, x_{13}, x_{15}, x_{14}, x_{16}]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{13}, x_{14}]$	$\leq_{lex}$	$[x_7, x_5, x_8, x_6, x_{15}, x_{13}, x_{16}, x_{14}, x_{11}, x_9, x_{12}, x_3, x_1]$
$[x_1, x_3, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}]$	$\leq_{lex}$	$[x_3, x_4, x_{11}, x_9, x_{12}, x_{10}, x_{15}, x_{13}, x_{16}, x_{14}, x_7, x_5]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_9, x_{10}, x_{11}, x_{12}]$	$\leq_{lex}$	$[x_{14}, x_{16}, x_{13}, x_{15}, x_6, x_8, x_5, x_2, x_4, x_1, x_3]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}]$	$\leq_{lex}$	$[x_{11}, x_9, x_{12}, x_{10}, x_7, x_5, x_8, x_{15}, x_{13}, x_{16}, x_{14}, x_3, x_1]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}]$	$\leq_{lex}$	$[x_7, x_5, x_8, x_6, x_{11}, x_9, x_{12}, x_{10}, x_3, x_1]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_9, x_{10}, x_{11}]$	$\leq_{lex}$	$[x_{14}, x_{16}, x_{13}, x_{15}, x_2, x_4, x_{10}, x_{12}, x_9]$
$[x_1, x_2, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}]$	$\leq_{lex}$	$[x_2, x_4, x_{14}, x_{16}, x_{13}, x_{15}, x_6, x_8, x_5, x_7]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}]$	$\leq_{lex}$	$[x_{11}, x_9, x_{12}, x_{10}, x_{15}, x_{13}, x_{16}, x_{14}, x_3, x_1]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}]$	$\leq_{lex}$	$[x_{15}, x_{13}, x_{16}, x_{14}, x_{11}, x_9, x_{12}, x_{10}, x_7, x_5]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9]$	$\leq_{lex}$	$[x_{10}, x_{12}, x_9, x_{11}, x_{14}, x_{16}, x_{13}, x_{15}, x_2]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{13}]$	$\leq_{lex}$	$[x_{14}, x_{16}, x_{13}, x_{15}, x_{10}, x_{12}, x_9, x_{11}, x_6, x_8, x_2]$
$[x_1, x_2, x_3, x_4, x_9, x_{10}, x_{11}, x_{12}]$	$\leq_{lex}$	$[x_8, x_6, x_7, x_5, x_{16}, x_{14}, x_{15}, x_{13}]$

Table A.12 – Continued on next page

Table A.12 – continued from previous page

$[x_1, x_2, x_3, x_4, x_9, x_{10}, x_{11}, x_{12}]$	$\leq_{lex}$	$[x_5, x_7, x_6, x_8, x_{13}, x_{15}, x_{14}, x_{16}]$
$[x_2, x_6, x_{10}, x_{14}]$	$\leq_{lex}$	$[x_3, x_7, x_{11}, x_{15}]$
$[x_1, x_2, x_3, x_4, x_5, x_9]$	$\leq_{lex}$	$[x_{15}, x_{14}, x_{13}, x_{16}, x_7, x_{11}]$
$[x_1, x_5, x_6, x_7, x_8, x_{13}]$	$\leq_{lex}$	$[x_3, x_{11}, x_{10}, x_9, x_{12}, x_{15}]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_9, x_{10}, x_{11}, x_{13}]$	$\leq_{lex}$	$[x_{14}, x_{15}, x_{16}, x_{13}, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_2]$
$[x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_9]$	$\leq_{lex}$	$[x_2, x_3, x_4, x_{10}, x_{11}, x_{12}, x_9, x_6]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_9, x_{10}, x_{11}]$	$\leq_{lex}$	$[x_{16}, x_{13}, x_{14}, x_{15}, x_8, x_5, x_6, x_{12}, x_9, x_{10}]$
$[x_1, x_2, x_3, x_4, x_6, x_{10}]$	$\leq_{lex}$	$[x_{13}, x_{16}, x_{15}, x_{14}, x_8, x_{12}]$
$[x_2, x_5, x_6, x_7, x_8, x_{14}]$	$\leq_{lex}$	$[x_4, x_9, x_{12}, x_{11}, x_{10}, x_{16}]$
$[x_1, x_2, x_3, x_4, x_5, x_{13}]$	$\leq_{lex}$	$[x_{11}, x_{10}, x_9, x_{12}, x_7, x_{15}]$
$[x_1, x_5, x_6, x_7, x_8, x_9]$	$\leq_{lex}$	$[x_3, x_{15}, x_{14}, x_{13}, x_{16}, x_{11}]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_9]$	$\leq_{lex}$	$[x_{10}, x_{11}, x_{12}, x_9, x_6, x_7, x_8, x_2]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}]$	$\leq_{lex}$	$[x_{14}, x_{15}, x_{16}, x_{13}, x_2, x_3, x_4, x_1, x_6, x_7, x_8, x_5]$
$[x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{13}]$	$\leq_{lex}$	$[x_2, x_3, x_4, x_{14}, x_{15}, x_{16}, x_{13}, x_{10}, x_{11}, x_{12}, x_6]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}]$	$\leq_{lex}$	$[x_8, x_5, x_6, x_7, x_{12}, x_9, x_{10}, x_{11}, x_{16}, x_{13}]$
$[x_1, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}]$	$\leq_{lex}$	$[x_4, x_{16}, x_{13}, x_{14}, x_{15}, x_{12}, x_9, x_{10}]$
$[x_1, x_2, x_3, x_4, x_6, x_{14}]$	$\leq_{lex}$	$[x_9, x_{12}, x_{11}, x_{10}, x_8, x_{16}]$
$[x_2, x_5, x_6, x_7, x_8, x_{10}]$	$\leq_{lex}$	$[x_4, x_{13}, x_{16}, x_{15}, x_{14}, x_{12}]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8]$	$\leq_{lex}$	$[x_{11}, x_{10}, x_9, x_{12}, x_{15}, x_{14}, x_{13}, x_{16}]$
$[x_1, x_2, x_3, x_4, x_9, x_{13}]$	$\leq_{lex}$	$[x_7, x_6, x_5, x_8, x_{11}, x_{15}]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8]$	$\leq_{lex}$	$[x_{15}, x_{14}, x_{13}, x_{16}, x_{11}, x_{10}, x_9, x_{12}]$
$[x_1, x_5, x_9, x_{10}, x_{11}, x_{12}]$	$\leq_{lex}$	$[x_3, x_7, x_{15}, x_{14}, x_{13}, x_{16}]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9]$	$\leq_{lex}$	$[x_{10}, x_{11}, x_{12}, x_9, x_{14}, x_{15}, x_{16}, x_{13}, x_6]$
$[x_1, x_2, x_3, x_4, x_5]$	$\leq_{lex}$	$[x_6, x_7, x_8, x_5, x_2]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9]$	$\leq_{lex}$	$[x_{14}, x_{15}, x_{16}, x_{13}, x_{10}, x_{11}, x_{12}, x_9, x_2]$
$[x_1, x_2, x_3, x_5, x_6, x_7, x_9, x_{10}, x_{11}, x_{12}, x_{13}]$	$\leq_{lex}$	$[x_2, x_3, x_4, x_6, x_7, x_8, x_{14}, x_{15}, x_{16}, x_{13}, x_{10}]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}]$	$\leq_{lex}$	$[x_{12}, x_9, x_{10}, x_{11}, x_{16}, x_{13}, x_{14}, x_{15}, x_8, x_5]$
$[x_1, x_5, x_9, x_{10}, x_{11}, x_{12}]$	$\leq_{lex}$	$[x_4, x_8, x_{16}, x_{13}, x_{14}, x_{15}]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8]$	$\leq_{lex}$	$[x_9, x_{12}, x_{11}, x_{10}, x_{13}, x_{16}, x_{15}, x_{14}]$
$[x_1, x_2, x_3, x_4, x_{10}, x_{14}]$	$\leq_{lex}$	$[x_5, x_8, x_7, x_6, x_{12}, x_{16}]$
$[x_1, x_2, x_3, x_4, x_6, x_7, x_8]$	$\leq_{lex}$	$[x_{13}, x_{16}, x_{15}, x_{14}, x_{12}, x_{11}, x_{10}]$
$[x_2, x_6, x_9, x_{10}, x_{11}, x_{12}]$	$\leq_{lex}$	$[x_4, x_8, x_{13}, x_{16}, x_{15}, x_{14}]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_9]$	$\leq_{lex}$	$[x_{10}, x_{11}, x_{12}, x_9, x_2, x_3, x_4, x_6]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_9, x_{10}, x_{11}, x_{12}, x_{13}]$	$\leq_{lex}$	$[x_{14}, x_{15}, x_{16}, x_{13}, x_6, x_7, x_8, x_2, x_3, x_4, x_1, x_{10}]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}]$	$\leq_{lex}$	$[x_8, x_5, x_6, x_7, x_{16}, x_{13}, x_{14}, x_{15}, x_{12}, x_9, x_{10}]$
$[x_1, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}]$	$\leq_{lex}$	$[x_4, x_{12}, x_9, x_{10}, x_{11}, x_{16}, x_{13}, x_{14}]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_9, x_{10}, x_{11}, x_{13}]$	$\leq_{lex}$	$[x_{14}, x_{15}, x_{16}, x_{13}, x_2, x_3, x_4, x_{10}, x_{11}, x_{12}, x_6]$
$[x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}]$	$\leq_{lex}$	$[x_2, x_3, x_4, x_{14}, x_{15}, x_{16}, x_{13}, x_6, x_7, x_8, x_5, x_{10}]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_9, x_{10}, x_{11}]$	$\leq_{lex}$	$[x_{12}, x_9, x_{10}, x_{11}, x_8, x_5, x_6, x_{16}, x_{13}, x_{14}]$
$[x_1, x_2, x_3, x_4, x_9, x_{10}, x_{11}, x_{12}]$	$\leq_{lex}$	$[x_7, x_6, x_5, x_8, x_{15}, x_{14}, x_{13}, x_{16}]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9]$	$\leq_{lex}$	$[x_{10}, x_{11}, x_{12}, x_9, x_{14}, x_{15}, x_{16}, x_{13}, x_2]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{13}]$	$\leq_{lex}$	$[x_{14}, x_{15}, x_{16}, x_{13}, x_{10}, x_{11}, x_{12}, x_9, x_6, x_7, x_2]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}]$	$\leq_{lex}$	$[x_{12}, x_9, x_{10}, x_{11}, x_{16}, x_{13}, x_{14}, x_{15}, x_4, x_1]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}]$	$\leq_{lex}$	$[x_{16}, x_{13}, x_{14}, x_{15}, x_{12}, x_9, x_{10}, x_{11}, x_8, x_5]$
$[x_1, x_2, x_3, x_4, x_9, x_{10}, x_{11}, x_{12}]$	$\leq_{lex}$	$[x_5, x_8, x_7, x_6, x_{13}, x_{16}, x_{15}, x_{14}]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7]$	$\leq_{lex}$	$[x_{11}, x_{12}, x_{10}, x_9, x_3, x_4, x_2]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_9, x_{10}, x_{11}, x_{13}]$	$\leq_{lex}$	$[x_{15}, x_{16}, x_{14}, x_{13}, x_7, x_8, x_6, x_{11}, x_{12}, x_{10}, x_3]$
$[x_1, x_2, x_5, x_6, x_7, x_8, x_9]$	$\leq_{lex}$	$[x_3, x_4, x_{11}, x_{12}, x_{10}, x_9, x_7]$

Table A.12 – Continued on next page

Table A.12 – continued from previous page

$[x_1, x_2, x_3, x_4, x_5, x_9]$	$\leq_{lex}$	$[x_{14}, x_{13}, x_{15}, x_{16}, x_6, x_{10}]$
$[x_1, x_5, x_6, x_7, x_8, x_{13}]$	$\leq_{lex}$	$[x_2, x_{10}, x_9, x_{11}, x_{12}, x_{14}]$
$[x_1, x_2, x_3, x_4, x_5, x_6]$	$\leq_{lex}$	$[x_8, x_7, x_5, x_6, x_{16}, x_{15}]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_9, x_{10}, x_{11}]$	$\leq_{lex}$	$[x_{16}, x_{15}, x_{13}, x_{14}, x_8, x_7, x_5, x_{12}, x_{11}, x_9]$
$[x_1, x_2, x_3, x_4, x_7, x_{11}]$	$\leq_{lex}$	$[x_{13}, x_{14}, x_{16}, x_{15}, x_8, x_{12}]$
$[x_3, x_5, x_6, x_7, x_8, x_{15}]$	$\leq_{lex}$	$[x_4, x_9, x_{10}, x_{12}, x_{11}, x_{16}]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_9]$	$\leq_{lex}$	$[x_{11}, x_{12}, x_{10}, x_9, x_7, x_8, x_6, x_3]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7]$	$\leq_{lex}$	$[x_{15}, x_{16}, x_{14}, x_{13}, x_3, x_4, x_2]$
$[x_1, x_2, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{13}]$	$\leq_{lex}$	$[x_3, x_4, x_{15}, x_{16}, x_{14}, x_{13}, x_{11}, x_{12}, x_{10}, x_7]$
$[x_1, x_2, x_3, x_4, x_5, x_{13}]$	$\leq_{lex}$	$[x_{10}, x_9, x_{11}, x_{12}, x_6, x_{14}]$
$[x_1, x_5, x_6, x_7, x_8, x_9]$	$\leq_{lex}$	$[x_2, x_{14}, x_{13}, x_{15}, x_{16}, x_{10}]$
$[x_1, x_2, x_3, x_4, x_5, x_6]$	$\leq_{lex}$	$[x_8, x_7, x_5, x_6, x_{12}, x_{11}]$
$[x_1, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}]$	$\leq_{lex}$	$[x_4, x_{16}, x_{15}, x_{13}, x_{14}, x_{12}, x_{11}, x_9]$
$[x_1, x_2, x_3, x_4, x_7, x_{15}]$	$\leq_{lex}$	$[x_9, x_{10}, x_{12}, x_{11}, x_8, x_{16}]$
$[x_3, x_5, x_6, x_7, x_8, x_{11}]$	$\leq_{lex}$	$[x_4, x_{13}, x_{14}, x_{16}, x_{15}, x_{12}]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9]$	$\leq_{lex}$	$[x_{11}, x_{12}, x_{10}, x_9, x_{15}, x_{16}, x_{14}, x_{13}, x_7]$
$[x_1, x_2, x_3, x_4, x_5]$	$\leq_{lex}$	$[x_7, x_8, x_6, x_5, x_3]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}]$	$\leq_{lex}$	$[x_{15}, x_{16}, x_{14}, x_{13}, x_{11}, x_{12}, x_{10}, x_9, x_3, x_4, x_2, x_1]$
$[x_1, x_2, x_5, x_6, x_9, x_{10}, x_{11}, x_{12}, x_{13}]$	$\leq_{lex}$	$[x_3, x_4, x_7, x_8, x_{15}, x_{16}, x_{14}, x_{13}, x_{11}]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8]$	$\leq_{lex}$	$[x_{10}, x_9, x_{11}, x_{12}, x_{14}, x_{13}, x_{15}, x_{16}]$
$[x_1, x_2, x_3, x_4, x_9, x_{13}]$	$\leq_{lex}$	$[x_6, x_5, x_7, x_8, x_{10}, x_{14}]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8]$	$\leq_{lex}$	$[x_{14}, x_{13}, x_{15}, x_{16}, x_{10}, x_9, x_{11}, x_{12}]$
$[x_1, x_5, x_9, x_{10}, x_{11}, x_{12}]$	$\leq_{lex}$	$[x_2, x_6, x_{14}, x_{13}, x_{15}, x_{16}]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}]$	$\leq_{lex}$	$[x_{12}, x_{11}, x_9, x_{10}, x_{16}, x_{15}, x_{13}, x_{14}, x_8, x_7, x_5]$
$[x_1, x_5, x_9, x_{10}, x_{11}, x_{12}]$	$\leq_{lex}$	$[x_4, x_8, x_{16}, x_{15}, x_{13}, x_{14}]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8]$	$\leq_{lex}$	$[x_9, x_{10}, x_{12}, x_{11}, x_{13}, x_{14}, x_{16}, x_{15}]$
$[x_1, x_2, x_3, x_4, x_{11}, x_{15}]$	$\leq_{lex}$	$[x_5, x_6, x_8, x_7, x_{12}, x_{16}]$
$[x_1, x_2, x_3, x_4, x_7, x_8]$	$\leq_{lex}$	$[x_{13}, x_{14}, x_{16}, x_{15}, x_{12}, x_{11}]$
$[x_3, x_7, x_9, x_{10}, x_{11}, x_{12}]$	$\leq_{lex}$	$[x_4, x_8, x_{13}, x_{14}, x_{16}, x_{15}]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_9, x_{10}, x_{11}]$	$\leq_{lex}$	$[x_{15}, x_{16}, x_{14}, x_{13}, x_7, x_8, x_6, x_3, x_4, x_2]$
$[x_1, x_5, x_6, x_7, x_8, x_9, x_{10}]$	$\leq_{lex}$	$[x_4, x_{12}, x_{11}, x_9, x_{10}, x_{16}, x_{15}]$
$[x_1, x_2, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}]$	$\leq_{lex}$	$[x_3, x_4, x_{15}, x_{16}, x_{14}, x_{13}, x_7, x_8, x_6]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_9, x_{10}]$	$\leq_{lex}$	$[x_{12}, x_{11}, x_9, x_{10}, x_8, x_7, x_5, x_{16}, x_{15}]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{11}, x_{13}]$	$\leq_{lex}$	$[x_{11}, x_{12}, x_{10}, x_9, x_{15}, x_{16}, x_{14}, x_{13}, x_3, x_2, x_7]$
$[x_1, x_2, x_3, x_4, x_7, x_9, x_{10}, x_{11}, x_{12}, x_{13}]$	$\leq_{lex}$	$[x_7, x_8, x_6, x_5, x_2, x_{15}, x_{16}, x_{14}, x_{13}, x_{11}]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{11}, x_{13}]$	$\leq_{lex}$	$[x_{15}, x_{16}, x_{14}, x_{13}, x_{11}, x_{12}, x_{10}, x_9, x_7, x_6, x_3]$
$[x_1, x_2, x_3, x_4, x_9, x_{10}, x_{11}, x_{12}]$	$\leq_{lex}$	$[x_6, x_5, x_7, x_8, x_{14}, x_{13}, x_{15}, x_{16}]$
$[x_1, x_5, x_9, x_{13}]$	$\leq_{lex}$	$[x_2, x_6, x_{10}, x_{14}]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{11}]$	$\leq_{lex}$	$[x_{12}, x_{11}, x_9, x_{10}, x_{16}, x_{15}, x_{13}, x_{14}, x_4, x_1]$
$[x_1, x_2, x_3, x_4, x_5, x_9, x_{10}, x_{11}, x_{12}]$	$\leq_{lex}$	$[x_8, x_7, x_5, x_6, x_4, x_{16}, x_{15}, x_{13}, x_{14}]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{11}]$	$\leq_{lex}$	$[x_{16}, x_{15}, x_{13}, x_{14}, x_{12}, x_{11}, x_9, x_{10}, x_8, x_5]$
$[x_1, x_2, x_3, x_4, x_9, x_{10}, x_{11}, x_{12}]$	$\leq_{lex}$	$[x_5, x_6, x_8, x_7, x_{13}, x_{14}, x_{16}, x_{15}]$
$[x_3, x_7, x_{11}, x_{15}]$	$\leq_{lex}$	$[x_4, x_8, x_{12}, x_{16}]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_9, x_{10}]$	$\leq_{lex}$	$[x_{15}, x_{13}, x_{14}, x_{16}, x_7, x_5, x_{11}, x_9]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8]$	$\leq_{lex}$	$[x_{10}, x_{12}, x_{11}, x_9, x_2, x_4, x_3, x_1]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_9, x_{10}, x_{13}]$	$\leq_{lex}$	$[x_{14}, x_{16}, x_{15}, x_{13}, x_6, x_8, x_{10}, x_{12}, x_2]$
$[x_1, x_2, x_5, x_6, x_7, x_8, x_9]$	$\leq_{lex}$	$[x_2, x_4, x_{10}, x_{12}, x_{11}, x_9, x_6]$
$[x_1, x_2, x_3, x_4, x_5, x_7, x_9, x_{11}]$	$\leq_{lex}$	$[x_{16}, x_{14}, x_{13}, x_{15}, x_8, x_5, x_{12}, x_9]$

Table A.12 – Continued on next page

Table A.12 – continued from previous page

$[x_1, x_2, x_3, x_4, x_6, x_7, x_8]$	$\leq_{lex}$	$[x_9, x_{11}, x_{12}, x_{10}, x_3, x_4, x_2]$
$[x_1, x_2, x_3, x_4, x_6, x_7, x_{10}, x_{11}, x_{14}]$	$\leq_{lex}$	$[x_{13}, x_{15}, x_{16}, x_{14}, x_7, x_8, x_{11}, x_{12}, x_3]$
$[x_2, x_3, x_5, x_6, x_7, x_8, x_{10}]$	$\leq_{lex}$	$[x_3, x_4, x_9, x_{11}, x_{12}, x_{10}, x_7]$
$[x_1, x_5, x_6, x_7, x_8, x_9, x_{10}]$	$\leq_{lex}$	$[x_3, x_{15}, x_{13}, x_{14}, x_{16}, x_{11}, x_9]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_9]$	$\leq_{lex}$	$[x_{10}, x_{12}, x_{11}, x_9, x_6, x_8, x_2]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8]$	$\leq_{lex}$	$[x_{14}, x_{16}, x_{15}, x_{13}, x_2, x_4, x_3, x_1]$
$[x_1, x_2, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{13}]$	$\leq_{lex}$	$[x_2, x_4, x_{14}, x_{16}, x_{15}, x_{13}, x_{10}, x_{12}, x_6]$
$[x_1, x_5, x_6, x_7, x_8, x_9, x_{11}]$	$\leq_{lex}$	$[x_4, x_{16}, x_{14}, x_{13}, x_{15}, x_{12}, x_9]$
$[x_1, x_2, x_3, x_4, x_6, x_7, x_{10}]$	$\leq_{lex}$	$[x_9, x_{11}, x_{12}, x_{10}, x_7, x_8, x_3]$
$[x_1, x_2, x_3, x_4, x_6, x_7, x_8]$	$\leq_{lex}$	$[x_{13}, x_{15}, x_{16}, x_{14}, x_3, x_4, x_2]$
$[x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{14}]$	$\leq_{lex}$	$[x_3, x_4, x_{13}, x_{15}, x_{16}, x_{14}, x_{11}, x_{12}, x_7]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}]$	$\leq_{lex}$	$[x_{11}, x_9, x_{10}, x_{12}, x_{15}, x_{13}, x_{14}, x_{16}, x_7, x_5]$
$[x_1, x_5, x_9, x_{10}, x_{11}, x_{12}]$	$\leq_{lex}$	$[x_3, x_7, x_{15}, x_{13}, x_{14}, x_{16}]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9]$	$\leq_{lex}$	$[x_{10}, x_{12}, x_{11}, x_9, x_{14}, x_{16}, x_{15}, x_{13}, x_6]$
$[x_1, x_2, x_3, x_4, x_5]$	$\leq_{lex}$	$[x_6, x_8, x_7, x_5, x_2]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9]$	$\leq_{lex}$	$[x_{14}, x_{16}, x_{15}, x_{13}, x_{10}, x_{12}, x_{11}, x_9, x_2]$
$[x_1, x_2, x_5, x_6, x_9, x_{10}, x_{11}, x_{12}, x_{13}]$	$\leq_{lex}$	$[x_2, x_4, x_6, x_8, x_{14}, x_{16}, x_{15}, x_{13}, x_{10}]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}]$	$\leq_{lex}$	$[x_{12}, x_{10}, x_9, x_{11}, x_{16}, x_{14}, x_{13}, x_{15}, x_8, x_6, x_5]$
$[x_1, x_5, x_9, x_{10}, x_{11}, x_{12}]$	$\leq_{lex}$	$[x_4, x_8, x_{16}, x_{14}, x_{13}, x_{15}]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_{10}]$	$\leq_{lex}$	$[x_9, x_{11}, x_{12}, x_{10}, x_{13}, x_{15}, x_{16}, x_{14}, x_7]$
$[x_1, x_2, x_3, x_4, x_6]$	$\leq_{lex}$	$[x_5, x_7, x_8, x_6, x_3]$
$[x_1, x_2, x_3, x_4, x_6, x_7, x_8, x_{10}]$	$\leq_{lex}$	$[x_{13}, x_{15}, x_{16}, x_{14}, x_{11}, x_{12}, x_{10}, x_3]$
$[x_2, x_3, x_6, x_7, x_9, x_{10}, x_{11}, x_{12}, x_{14}]$	$\leq_{lex}$	$[x_3, x_4, x_7, x_8, x_{13}, x_{15}, x_{16}, x_{14}, x_{11}]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}]$	$\leq_{lex}$	$[x_7, x_5, x_6, x_8, x_{15}, x_{13}, x_{14}, x_{16}, x_{11}, x_9]$
$[x_1, x_5, x_6, x_7, x_8, x_9, x_{10}]$	$\leq_{lex}$	$[x_3, x_{11}, x_9, x_{10}, x_{12}, x_{15}, x_{13}]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_{13}]$	$\leq_{lex}$	$[x_{10}, x_{12}, x_{11}, x_9, x_2, x_4, x_3, x_{14}]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_9, x_{10}, x_{11}, x_{12}]$	$\leq_{lex}$	$[x_{14}, x_{16}, x_{15}, x_{13}, x_6, x_8, x_2, x_4, x_3, x_1]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{11}]$	$\leq_{lex}$	$[x_8, x_6, x_5, x_7, x_{16}, x_{14}, x_{13}, x_{15}, x_{12}, x_9]$
$[x_1, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}]$	$\leq_{lex}$	$[x_4, x_{12}, x_{10}, x_9, x_{11}, x_{16}, x_{14}, x_{13}]$
$[x_1, x_2, x_3, x_4, x_6, x_7, x_{14}]$	$\leq_{lex}$	$[x_9, x_{11}, x_{12}, x_{10}, x_3, x_4, x_{15}]$
$[x_1, x_2, x_3, x_4, x_6, x_7, x_{10}, x_{11}, x_{12}]$	$\leq_{lex}$	$[x_{13}, x_{15}, x_{16}, x_{14}, x_7, x_8, x_3, x_4, x_2]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_9, x_{10}]$	$\leq_{lex}$	$[x_{11}, x_9, x_{10}, x_{12}, x_7, x_5, x_{15}, x_{13}]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_{13}, x_{14}]$	$\leq_{lex}$	$[x_7, x_5, x_6, x_8, x_{11}, x_9, x_{10}, x_{12}, x_{15}, x_{13}]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_9]$	$\leq_{lex}$	$[x_{14}, x_{16}, x_{15}, x_{13}, x_2, x_4, x_3, x_{10}]$
$[x_1, x_2, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}]$	$\leq_{lex}$	$[x_2, x_4, x_{14}, x_{16}, x_{15}, x_{13}, x_6, x_8, x_7, x_5]$
$[x_1, x_2, x_3, x_4, x_5, x_7, x_9, x_{10}, x_{11}]$	$\leq_{lex}$	$[x_{12}, x_{10}, x_9, x_{11}, x_8, x_5, x_{16}, x_{14}, x_{13}]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_{13}, x_{15}]$	$\leq_{lex}$	$[x_8, x_6, x_5, x_7, x_{12}, x_{10}, x_9, x_{11}, x_{16}, x_{13}]$
$[x_1, x_2, x_3, x_4, x_6, x_7, x_{10}]$	$\leq_{lex}$	$[x_{13}, x_{15}, x_{16}, x_{14}, x_3, x_4, x_{11}]$
$[x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}]$	$\leq_{lex}$	$[x_3, x_4, x_{13}, x_{15}, x_{16}, x_{14}, x_7, x_8, x_6]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}]$	$\leq_{lex}$	$[x_{11}, x_9, x_{10}, x_{12}, x_{15}, x_{13}, x_{14}, x_{16}, x_3, x_1]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}]$	$\leq_{lex}$	$[x_{15}, x_{13}, x_{14}, x_{16}, x_{11}, x_9, x_{10}, x_{12}, x_7, x_5]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9]$	$\leq_{lex}$	$[x_{10}, x_{12}, x_{11}, x_9, x_{14}, x_{16}, x_{15}, x_{13}, x_2]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9]$	$\leq_{lex}$	$[x_{14}, x_{16}, x_{15}, x_{13}, x_{10}, x_{12}, x_{11}, x_9, x_6]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{11}]$	$\leq_{lex}$	$[x_{12}, x_{10}, x_9, x_{11}, x_{16}, x_{14}, x_{13}, x_{15}, x_4, x_1]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{11}]$	$\leq_{lex}$	$[x_{16}, x_{14}, x_{13}, x_{15}, x_{12}, x_{10}, x_9, x_{11}, x_8, x_5]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_{10}]$	$\leq_{lex}$	$[x_9, x_{11}, x_{12}, x_{10}, x_{13}, x_{15}, x_{16}, x_{14}, x_3]$
$[x_1, x_2, x_3, x_4, x_6, x_7, x_8, x_{10}]$	$\leq_{lex}$	$[x_{13}, x_{15}, x_{16}, x_{14}, x_{11}, x_{12}, x_{10}, x_7]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8]$	$\leq_{lex}$	$[x_{11}, x_{10}, x_{12}, x_9, x_3, x_2, x_4, x_1]$

Table A.12 – Continued on next page

Table A.12 – continued from previous page

$$[x_1, x_2, x_3, x_4, x_5, x_7, x_9, x_{11}, x_{13}] \leq_{lex} [x_{15}, x_{14}, x_{16}, x_{13}, x_7, x_8, x_{11}, x_{12}, x_3]$$
$$[x_1, x_3, x_5, x_6, x_7, x_8, x_9] \leq_{lex} [x_3, x_4, x_{11}, x_{10}, x_{12}, x_9, x_7]$$
$$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8] \leq_{lex} [x_{10}, x_{11}, x_9, x_{12}, x_2, x_3, x_1, x_4]$$
$$[x_1, x_2, x_3, x_4, x_5, x_6, x_9, x_{10}, x_{13}] \leq_{lex} [x_{14}, x_{15}, x_{13}, x_{16}, x_6, x_7, x_{10}, x_{11}, x_2]$$
$$[x_1, x_2, x_5, x_6, x_7, x_8, x_9] \leq_{lex} [x_2, x_3, x_{10}, x_{11}, x_9, x_{12}, x_6]$$
$$[x_1, x_2, x_3, x_4, x_5, x_6, x_9, x_{10}] \leq_{lex} [x_{16}, x_{13}, x_{15}, x_{14}, x_8, x_5, x_{12}, x_9]$$
$$[x_1, x_2, x_3, x_4, x_6, x_7, x_{10}, x_{11}] \leq_{lex} [x_{13}, x_{16}, x_{14}, x_{15}, x_8, x_6, x_{12}, x_{10}]$$
$$[x_1, x_2, x_3, x_4, x_5, x_7, x_9] \leq_{lex} [x_{11}, x_{10}, x_{12}, x_9, x_7, x_8, x_3]$$
$$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8] \leq_{lex} [x_{15}, x_{14}, x_{16}, x_{13}, x_3, x_2, x_4, x_1]$$
$$[x_1, x_3, x_5, x_6, x_7, x_8, x_9, x_{11}, x_{13}] \leq_{lex} [x_3, x_4, x_{15}, x_{14}, x_{16}, x_{13}, x_{11}, x_{12}, x_7]$$
$$[x_1, x_2, x_3, x_4, x_5, x_6, x_9] \leq_{lex} [x_{10}, x_{11}, x_9, x_{12}, x_6, x_7, x_2]$$
$$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8] \leq_{lex} [x_{14}, x_{15}, x_{13}, x_{16}, x_2, x_3, x_1, x_4]$$
$$[x_1, x_2, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{13}] \leq_{lex} [x_2, x_3, x_{14}, x_{15}, x_{13}, x_{16}, x_{10}, x_{11}, x_6]$$
$$[x_1, x_5, x_6, x_7, x_8, x_9, x_{10}] \leq_{lex} [x_4, x_{16}, x_{13}, x_{15}, x_{14}, x_{12}, x_9]$$
$$[x_2, x_5, x_6, x_7, x_8, x_{10}, x_{11}] \leq_{lex} [x_4, x_{13}, x_{16}, x_{14}, x_{15}, x_{12}, x_{10}]$$
$$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}] \leq_{lex} [x_{11}, x_{10}, x_{12}, x_9, x_{15}, x_{14}, x_{16}, x_{13}, x_7, x_6]$$
$$[x_1, x_2, x_3, x_4, x_5] \leq_{lex} [x_7, x_6, x_8, x_5, x_3]$$
$$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}] \leq_{lex} [x_{15}, x_{14}, x_{16}, x_{13}, x_{11}, x_{10}, x_{12}, x_9, x_3, x_2]$$
$$[x_1, x_3, x_5, x_7, x_9, x_{10}, x_{11}, x_{12}, x_{13}] \leq_{lex} [x_3, x_4, x_7, x_8, x_{15}, x_{14}, x_{16}, x_{13}, x_{11}]$$
$$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9] \leq_{lex} [x_{10}, x_{11}, x_9, x_{12}, x_{14}, x_{15}, x_{13}, x_{16}, x_6]$$
$$[x_1, x_2, x_3, x_4, x_5] \leq_{lex} [x_6, x_7, x_5, x_8, x_2]$$
$$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9] \leq_{lex} [x_{14}, x_{15}, x_{13}, x_{16}, x_{10}, x_{11}, x_9, x_{12}, x_2]$$
$$[x_1, x_2, x_5, x_6, x_9, x_{10}, x_{11}, x_{12}, x_{13}] \leq_{lex} [x_2, x_3, x_6, x_7, x_{14}, x_{15}, x_{13}, x_{16}, x_{10}]$$
$$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}] \leq_{lex} [x_{12}, x_9, x_{11}, x_{10}, x_{16}, x_{13}, x_{15}, x_{14}, x_8, x_5]$$
$$[x_1, x_5, x_9, x_{10}, x_{11}, x_{12}] \leq_{lex} [x_4, x_8, x_{16}, x_{13}, x_{15}, x_{14}]$$
$$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_{10}, x_{11}] \leq_{lex} [x_9, x_{12}, x_{10}, x_{11}, x_{13}, x_{16}, x_{14}, x_{15}, x_8, x_6]$$
$$[x_2, x_6, x_9, x_{10}, x_{11}, x_{12}] \leq_{lex} [x_4, x_8, x_{13}, x_{16}, x_{14}, x_{15}]$$
$$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_{13}] \leq_{lex} [x_{11}, x_{10}, x_{12}, x_9, x_3, x_2, x_4, x_{15}]$$
$$[x_1, x_2, x_3, x_4, x_5, x_7, x_9, x_{10}, x_{11}, x_{12}] \leq_{lex} [x_{15}, x_{14}, x_{16}, x_{13}, x_7, x_8, x_3, x_2, x_4, x_1]$$
$$[x_1, x_2, x_3, x_4, x_5, x_6, x_{13}] \leq_{lex} [x_{10}, x_{11}, x_9, x_{12}, x_2, x_3, x_{14}]$$
$$[x_1, x_2, x_3, x_4, x_5, x_6, x_9, x_{10}, x_{11}, x_{12}] \leq_{lex} [x_{14}, x_{15}, x_{13}, x_{16}, x_6, x_7, x_2, x_3, x_1, x_4]$$
$$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}] \leq_{lex} [x_8, x_5, x_7, x_6, x_{16}, x_{13}, x_{15}, x_{14}, x_{12}, x_9]$$
$$[x_1, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}] \leq_{lex} [x_4, x_{12}, x_9, x_{11}, x_{10}, x_{16}, x_{13}, x_{15}]$$
$$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_{10}, x_{11}] \leq_{lex} [x_5, x_8, x_6, x_7, x_{13}, x_{16}, x_{14}, x_{15}, x_{12}, x_{10}]$$
$$[x_2, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}] \leq_{lex} [x_4, x_9, x_{12}, x_{10}, x_{11}, x_{13}, x_{16}, x_{14}]$$
$$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_9] \leq_{lex} [x_{15}, x_{14}, x_{16}, x_{13}, x_3, x_2, x_4, x_{11}]$$
$$[x_1, x_3, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}] \leq_{lex} [x_3, x_4, x_{15}, x_{14}, x_{16}, x_{13}, x_7, x_6, x_8, x_5]$$
$$[x_1, x_2, x_3, x_4, x_5, x_6, x_9] \leq_{lex} [x_{14}, x_{15}, x_{13}, x_{16}, x_2, x_3, x_{10}]$$
$$[x_1, x_2, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}] \leq_{lex} [x_2, x_3, x_{14}, x_{15}, x_{13}, x_{16}, x_6, x_7, x_5, x_8]$$
$$[x_1, x_2, x_3, x_4, x_5, x_6, x_9, x_{10}, x_{11}] \leq_{lex} [x_{12}, x_9, x_{11}, x_{10}, x_8, x_5, x_{16}, x_{13}, x_{15}]$$
$$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_{13}, x_{14}] \leq_{lex} [x_8, x_5, x_7, x_6, x_{12}, x_9, x_{11}, x_{10}, x_{16}, x_{13}]$$
$$[x_1, x_2, x_3, x_4, x_6, x_7, x_9, x_{10}, x_{11}] \leq_{lex} [x_9, x_{12}, x_{10}, x_{11}, x_8, x_6, x_{13}, x_{16}, x_{14}]$$
$$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_{14}, x_{15}] \leq_{lex} [x_5, x_8, x_6, x_7, x_9, x_{12}, x_{10}, x_{11}, x_{16}, x_{14}]$$
$$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9] \leq_{lex} [x_{11}, x_{10}, x_{12}, x_9, x_{15}, x_{14}, x_{16}, x_{13}, x_3]$$
$$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9] \leq_{lex} [x_{15}, x_{14}, x_{16}, x_{13}, x_{11}, x_{10}, x_{12}, x_9, x_7]$$
$$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9] \leq_{lex} [x_{10}, x_{11}, x_9, x_{12}, x_{14}, x_{15}, x_{13}, x_{16}, x_2]$$
$$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9] \leq_{lex} [x_{14}, x_{15}, x_{13}, x_{16}, x_{10}, x_{11}, x_9, x_{12}, x_6]$$
$$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}] \leq_{lex} [x_{12}, x_9, x_{11}, x_{10}, x_{16}, x_{13}, x_{15}, x_{14}, x_4, x_1]$$

Table A.12 – Continued on next page

Table A.12 – continued from previous page

$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}]$	$\leq_{lex}$	$[x_{16}, x_{13}, x_{15}, x_{14}, x_{12}, x_9, x_{11}, x_{10}, x_8, x_5]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_{10}, x_{11}]$	$\leq_{lex}$	$[x_9, x_{12}, x_{10}, x_{11}, x_{13}, x_{16}, x_{14}, x_{15}, x_4, x_2]$
$[x_1, x_2, x_3, x_4, x_6, x_7, x_8, x_{10}, x_{11}]$	$\leq_{lex}$	$[x_{13}, x_{16}, x_{14}, x_{15}, x_{12}, x_{10}, x_{11}, x_8, x_6]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_9, x_{10}]$	$\leq_{lex}$	$[x_{15}, x_{16}, x_{13}, x_{14}, x_7, x_8, x_{11}, x_{12}]$
$[x_1, x_2, x_5, x_6, x_7, x_8, x_{13}, x_{14}]$	$\leq_{lex}$	$[x_3, x_4, x_{11}, x_{12}, x_9, x_{10}, x_{15}, x_{16}]$
$[x_1, x_2, x_3, x_4, x_5, x_7, x_9, x_{11}]$	$\leq_{lex}$	$[x_{14}, x_{13}, x_{16}, x_{15}, x_6, x_8, x_{10}, x_{12}]$
$[x_1, x_3, x_5, x_6, x_7, x_8, x_{13}, x_{15}]$	$\leq_{lex}$	$[x_2, x_4, x_{10}, x_9, x_{12}, x_{11}, x_{14}, x_{16}]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_9, x_{10}]$	$\leq_{lex}$	$[x_{16}, x_{15}, x_{14}, x_{13}, x_8, x_7, x_{12}, x_{11}]$
$[x_1, x_5, x_6, x_7, x_8, x_{13}, x_{14}]$	$\leq_{lex}$	$[x_4, x_{12}, x_{11}, x_{10}, x_9, x_{16}, x_{15}]$
$[x_1, x_2, x_3, x_4]$	$\leq_{lex}$	$[x_5, x_6, x_7, x_8]$
$[x_5, x_6, x_7, x_8]$	$\leq_{lex}$	$[x_9, x_{10}, x_{11}, x_{12}]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_{13}, x_{14}]$	$\leq_{lex}$	$[x_{11}, x_{12}, x_9, x_{10}, x_7, x_8, x_{15}, x_{16}]$
$[x_1, x_2, x_5, x_6, x_7, x_8, x_9, x_{10}]$	$\leq_{lex}$	$[x_3, x_4, x_{15}, x_{16}, x_{13}, x_{14}, x_{11}, x_{12}]$
$[x_1, x_2, x_3, x_4, x_5, x_7, x_{13}, x_{15}]$	$\leq_{lex}$	$[x_{10}, x_9, x_{12}, x_{11}, x_6, x_8, x_{14}, x_{16}]$
$[x_1, x_3, x_5, x_6, x_7, x_8, x_9, x_{11}]$	$\leq_{lex}$	$[x_2, x_4, x_{14}, x_{13}, x_{16}, x_{15}, x_{10}, x_{12}]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_{13}, x_{14}]$	$\leq_{lex}$	$[x_{12}, x_{11}, x_{10}, x_9, x_8, x_7, x_{16}, x_{15}]$
$[x_1, x_5, x_6, x_7, x_8, x_9, x_{10}]$	$\leq_{lex}$	$[x_4, x_{16}, x_{15}, x_{14}, x_{13}, x_{12}, x_{11}]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8]$	$\leq_{lex}$	$[x_{11}, x_{12}, x_9, x_{10}, x_{15}, x_{16}, x_{13}, x_{14}]$
$[x_1, x_2, x_3, x_4, x_9, x_{10}, x_{13}, x_{14}]$	$\leq_{lex}$	$[x_7, x_8, x_5, x_6, x_{11}, x_{12}, x_{15}, x_{16}]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8]$	$\leq_{lex}$	$[x_{15}, x_{16}, x_{13}, x_{14}, x_{11}, x_{12}, x_9, x_{10}]$
$[x_1, x_2, x_5, x_6, x_9, x_{10}, x_{11}, x_{12}]$	$\leq_{lex}$	$[x_3, x_4, x_7, x_8, x_{15}, x_{16}, x_{13}, x_{14}]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8]$	$\leq_{lex}$	$[x_{10}, x_9, x_{12}, x_{11}, x_{14}, x_{13}, x_{16}, x_{15}]$
$[x_1, x_2, x_3, x_4, x_9, x_{11}, x_{13}, x_{15}]$	$\leq_{lex}$	$[x_6, x_5, x_8, x_7, x_{10}, x_{12}, x_{14}, x_{16}]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8]$	$\leq_{lex}$	$[x_{14}, x_{13}, x_{16}, x_{15}, x_{10}, x_9, x_{12}, x_{11}]$
$[x_1, x_3, x_5, x_7, x_9, x_{10}, x_{11}, x_{12}]$	$\leq_{lex}$	$[x_2, x_4, x_6, x_8, x_{14}, x_{13}, x_{16}, x_{15}]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8]$	$\leq_{lex}$	$[x_{12}, x_{11}, x_{10}, x_9, x_{16}, x_{15}, x_{14}, x_{13}]$
$[x_1, x_2, x_3, x_4, x_9, x_{10}, x_{13}, x_{14}]$	$\leq_{lex}$	$[x_8, x_7, x_6, x_5, x_{12}, x_{11}, x_{16}, x_{15}]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8]$	$\leq_{lex}$	$[x_{16}, x_{15}, x_{14}, x_{13}, x_{12}, x_{11}, x_{10}, x_9]$
$[x_1, x_5, x_9, x_{10}, x_{11}, x_{12}]$	$\leq_{lex}$	$[x_4, x_8, x_{16}, x_{15}, x_{14}, x_{13}]$
$[x_9, x_{10}, x_{11}, x_{12}]$	$\leq_{lex}$	$[x_{13}, x_{14}, x_{15}, x_{16}]$
$[x_1, x_2, x_3, x_4, x_9, x_{10}, x_{11}, x_{12}]$	$\leq_{lex}$	$[x_7, x_8, x_5, x_6, x_{15}, x_{16}, x_{13}, x_{14}]$
$[x_1, x_2, x_3, x_4, x_9, x_{10}, x_{11}, x_{12}]$	$\leq_{lex}$	$[x_6, x_5, x_8, x_7, x_{14}, x_{13}, x_{16}, x_{15}]$
$[x_1, x_2, x_3, x_4, x_9, x_{10}, x_{11}, x_{12}]$	$\leq_{lex}$	$[x_8, x_7, x_6, x_5, x_{16}, x_{15}, x_{14}, x_{13}]$

Table A.13: Approximate minimal set of *lex*-constraints, $M_{4\times4}$, domain 2

$[x_9, x_{10}, x_{11}, x_{12}]$	$\leq_{lex}$	$[x_{13}, x_{14}, x_{15}, x_{16}]$
$[x_1, x_3, x_5, x_6, x_7, x_8, x_{13}, x_{15}]$	$\leq_{lex}$	$[x_2, x_4, x_{10}, x_9, x_{12}, x_{11}, x_{14}, x_{16}]$
$[x_3, x_7, x_{11}, x_{15}]$	$\leq_{lex}$	$[x_4, x_8, x_{12}, x_{16}]$
$[x_1, x_5, x_9, x_{13}]$	$\leq_{lex}$	$[x_2, x_6, x_{10}, x_{14}]$
$[x_1, x_2, x_3, x_4, x_{11}, x_{15}]$	$\leq_{lex}$	$[x_5, x_6, x_8, x_7, x_{12}, x_{16}]$
$[x_1, x_5, x_9, x_{10}, x_{11}, x_{12}]$	$\leq_{lex}$	$[x_2, x_6, x_{14}, x_{13}, x_{15}, x_{16}]$
$[x_3, x_5, x_6, x_7, x_8, x_{15}]$	$\leq_{lex}$	$[x_4, x_9, x_{10}, x_{12}, x_{11}, x_{16}]$
$[x_1, x_5, x_6, x_7, x_8, x_{13}]$	$\leq_{lex}$	$[x_2, x_{10}, x_9, x_{11}, x_{12}, x_{14}]$
$[x_2, x_6, x_{10}, x_{14}]$	$\leq_{lex}$	$[x_3, x_7, x_{11}, x_{15}]$
$[x_2, x_6, x_9, x_{10}, x_{11}, x_{12}]$	$\leq_{lex}$	$[x_3, x_7, x_{13}, x_{15}, x_{14}, x_{16}]$

Table A.13 – Continued on next page

Table A.13 – continued from previous page

$[x_1, x_2, x_3, x_4, x_{10}, x_{14}]$	$\leq_{lex}$	$[x_5, x_7, x_6, x_8, x_{11}, x_{15}]$
$[x_2, x_5, x_6, x_7, x_8, x_{14}]$	$\leq_{lex}$	$[x_3, x_9, x_{11}, x_{10}, x_{12}, x_{15}]$
$[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}]$	$\leq_{lex}$	$[x_7, x_5, x_8, x_6, x_{15}, x_{13}, x_{16}, x_{14}, x_3, x_1, x_4, x_2]$
$[x_1, x_2, x_3, x_4, x_9, x_{10}, x_{11}, x_{12}]$	$\leq_{lex}$	$[x_5, x_7, x_6, x_8, x_{13}, x_{15}, x_{14}, x_{16}]$

Table A.14: Minimized DNF of lex-constraints for $M_{4\times 4}$, domain 2

$(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \neg x_5 \wedge \neg x_6 \wedge \neg x_7 \wedge x_8 \wedge \neg x_9 \wedge \neg x_{10} \wedge x_{12} \wedge x_{14} \wedge x_{15}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \neg x_5 \wedge \neg x_6 \wedge x_8 \wedge \neg x_9 \wedge \neg x_{10} \wedge x_{11} \wedge x_{12} \wedge x_{14} \wedge x_{16}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge x_4 \wedge \neg x_5 \wedge \neg x_6 \wedge x_7 \wedge x_8 \wedge \neg x_9 \wedge x_{10} \wedge x_{11} \wedge x_{13} \wedge x_{14}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge x_4 \wedge \neg x_5 \wedge x_6 \wedge x_7 \wedge \neg x_8 \wedge \neg x_9 \wedge x_{10} \wedge x_{11} \wedge x_{13} \wedge x_{15}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge x_4 \wedge \neg x_5 \wedge \neg x_6 \wedge x_7 \wedge \neg x_9 \wedge x_{10} \wedge \neg x_{11} \wedge x_{12} \wedge x_{13} \wedge x_{15}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \neg x_5 \wedge \neg x_6 \wedge x_7 \wedge x_8 \wedge \neg x_9 \wedge x_{10} \wedge \neg x_{11} \wedge x_{12} \wedge x_{13} \wedge x_{15}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge x_4 \wedge \neg x_5 \wedge \neg x_6 \wedge x_7 \wedge x_8 \wedge \neg x_9 \wedge x_{10} \wedge \neg x_{11} \wedge x_{12} \wedge x_{13} \wedge x_{15}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge x_4 \wedge \neg x_5 \wedge \neg x_6 \wedge x_8 \wedge \neg x_9 \wedge x_{10} \wedge x_{11} \wedge \neg x_{12} \wedge x_{13} \wedge x_{15}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \neg x_5 \wedge \neg x_6 \wedge \neg x_7 \wedge x_8 \wedge \neg x_9 \wedge x_{10} \wedge x_{11} \wedge \neg x_{12} \wedge x_{13} \wedge x_{15}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \neg x_4 \wedge \neg x_5 \wedge \neg x_6 \wedge \neg x_7 \wedge \neg x_9 \wedge \neg x_{10} \wedge \neg x_{11} \wedge x_{12} \wedge \neg x_{13} \wedge x_{15}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \neg x_5 \wedge \neg x_6 \wedge \neg x_7 \wedge x_8 \wedge \neg x_9 \wedge \neg x_{10} \wedge \neg x_{11} \wedge x_{12} \wedge \neg x_{13} \wedge x_{15}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge x_4 \wedge \neg x_5 \wedge \neg x_6 \wedge x_8 \wedge \neg x_9 \wedge \neg x_{10} \wedge x_{11} \wedge x_{12} \wedge x_{14} \wedge x_{15}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge x_4 \wedge \neg x_5 \wedge \neg x_6 \wedge x_7 \wedge \neg x_9 \wedge x_{10} \wedge \neg x_{11} \wedge x_{12} \wedge x_{14} \wedge x_{15}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \neg x_5 \wedge \neg x_6 \wedge x_7 \wedge x_8 \wedge \neg x_9 \wedge x_{10} \wedge \neg x_{11} \wedge x_{12} \wedge x_{14} \wedge x_{15}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge x_4 \wedge \neg x_5 \wedge \neg x_6 \wedge x_7 \wedge x_8 \wedge \neg x_9 \wedge x_{10} \wedge \neg x_{11} \wedge x_{12} \wedge x_{14} \wedge x_{15}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \neg x_4 \wedge \neg x_5 \wedge \neg x_6 \wedge \neg x_7 \wedge \neg x_9 \wedge \neg x_{10} \wedge \neg x_{11} \wedge x_{12} \wedge x_{14} \wedge x_{15}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge x_4 \wedge \neg x_5 \wedge \neg x_6 \wedge x_8 \wedge \neg x_9 \wedge x_{10} \wedge x_{11} \wedge \neg x_{12} \wedge x_{14} \wedge x_{15}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \neg x_5 \wedge \neg x_6 \wedge \neg x_7 \wedge x_8 \wedge \neg x_9 \wedge x_{10} \wedge x_{11} \wedge \neg x_{12} \wedge x_{14} \wedge x_{15}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge x_4 \wedge \neg x_5 \wedge \neg x_6 \wedge x_7 \wedge x_{10} \wedge \neg x_{11} \wedge x_{12} \wedge x_{13} \wedge x_{14} \wedge x_{15}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \neg x_5 \wedge \neg x_6 \wedge x_7 \wedge x_8 \wedge x_{10} \wedge \neg x_{11} \wedge x_{12} \wedge x_{13} \wedge x_{14} \wedge x_{15}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \neg x_5 \wedge \neg x_6 \wedge \neg x_7 \wedge x_8 \wedge x_{10} \wedge x_{11} \wedge \neg x_{12} \wedge x_{13} \wedge x_{14} \wedge x_{15}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \neg x_5 \wedge \neg x_6 \wedge \neg x_7 \wedge x_8 \wedge \neg x_9 \wedge \neg x_{10} \wedge x_{11} \wedge \neg x_{12} \wedge x_{14} \wedge \neg x_{15}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \neg x_5 \wedge \neg x_6 \wedge x_7 \wedge x_8 \wedge \neg x_9 \wedge x_{11} \wedge x_{12} \wedge x_{13} \wedge x_{14} \wedge \neg x_{15}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge x_4 \wedge \neg x_5 \wedge \neg x_6 \wedge x_7 \wedge x_8 \wedge \neg x_9 \wedge x_{11} \wedge x_{12} \wedge x_{13} \wedge x_{14} \wedge \neg x_{15}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \neg x_4 \wedge \neg x_5 \wedge \neg x_6 \wedge \neg x_7 \wedge \neg x_9 \wedge \neg x_{10} \wedge x_{11} \wedge x_{12} \wedge x_{14} \wedge x_{16}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge x_4 \wedge \neg x_5 \wedge \neg x_6 \wedge x_7 \wedge x_8 \wedge \neg x_9 \wedge \neg x_{10} \wedge x_{11} \wedge x_{12} \wedge x_{14} \wedge x_{16}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge x_3 \wedge x_4 \wedge \neg x_5 \wedge x_7 \wedge x_8 \wedge x_9 \wedge x_{10} \wedge \neg x_{11} \wedge x_{13} \wedge x_{14} \wedge x_{16}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \neg x_5 \wedge x_7 \wedge x_8 \wedge \neg x_9 \wedge x_{10} \wedge x_{11} \wedge x_{12} \wedge x_{13} \wedge x_{15} \wedge x_{16}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \neg x_5 \wedge \neg x_6 \wedge \neg x_7 \wedge x_8 \wedge \neg x_9 \wedge \neg x_{10} \wedge x_{11} \wedge \neg x_{13} \wedge x_{15} \wedge x_{16}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \neg x_4 \wedge \neg x_5 \wedge \neg x_6 \wedge \neg x_7 \wedge \neg x_9 \wedge \neg x_{10} \wedge x_{12} \wedge \neg x_{13} \wedge x_{15} \wedge x_{16}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \neg x_5 \wedge \neg x_6 \wedge x_8 \wedge \neg x_9 \wedge \neg x_{10} \wedge x_{11} \wedge x_{12} \wedge \neg x_{13} \wedge x_{15} \wedge x_{16}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \neg x_4 \wedge \neg x_5 \wedge \neg x_6 \wedge \neg x_7 \wedge \neg x_9 \wedge x_{11} \wedge x_{12} \wedge x_{14} \wedge x_{15} \wedge x_{16}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \neg x_5 \wedge x_7 \wedge x_8 \wedge \neg x_9 \wedge x_{10} \wedge x_{11} \wedge x_{12} \wedge x_{14} \wedge x_{15} \wedge x_{16}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge x_4 \wedge \neg x_5 \wedge x_6 \wedge x_7 \wedge \neg x_8 \wedge x_9 \wedge \neg x_{10} \wedge x_{11} \wedge \neg x_{12} \wedge x_{13} \wedge x_{14}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge x_3 \wedge x_4 \wedge \neg x_5 \wedge x_6 \wedge \neg x_7 \wedge x_8 \wedge \neg x_9 \wedge x_{10} \wedge x_{11} \wedge x_{12} \wedge x_{13} \wedge x_{15}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge x_4 \wedge \neg x_5 \wedge \neg x_6 \wedge \neg x_7 \wedge x_8 \wedge \neg x_9 \wedge \neg x_{10} \wedge x_{11} \wedge \neg x_{12} \wedge \neg x_{13} \wedge x_{15}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge x_4 \wedge \neg x_5 \wedge x_6 \wedge x_7 \wedge \neg x_8 \wedge \neg x_9 \wedge x_{10} \wedge x_{11} \wedge \neg x_{12} \wedge x_{14} \wedge x_{15}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge x_3 \wedge x_4 \wedge \neg x_5 \wedge x_6 \wedge \neg x_7 \wedge x_8 \wedge x_9 \wedge \neg x_{10} \wedge x_{12} \wedge x_{13} \wedge x_{14} \wedge x_{15}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge x_4 \wedge \neg x_5 \wedge x_6 \wedge x_7 \wedge x_9 \wedge \neg x_{10} \wedge x_{11} \wedge x_{12} \wedge x_{13} \wedge x_{14} \wedge x_{15}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge x_3 \wedge x_4 \wedge \neg x_5 \wedge x_7 \wedge x_8 \wedge x_9 \wedge x_{10} \wedge \neg x_{11} \wedge x_{12} \wedge x_{13} \wedge x_{14} \wedge x_{15}) \vee$

Table A.14 – Continued on next page

Table A.14 – continued from previous page

$(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge x_4 \wedge x_6 \wedge x_7 \wedge \neg x_8 \wedge x_9 \wedge x_{10} \wedge x_{11} \wedge \neg x_{12} \wedge x_{13} \wedge x_{14} \wedge x_{15}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge x_4 \wedge \neg x_5 \wedge \neg x_6 \wedge x_7 \wedge \neg x_8 \wedge \neg x_9 \wedge \neg x_{10} \wedge x_{11} \wedge x_{12} \wedge x_{14} \wedge \neg x_{15}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge x_3 \wedge x_4 \wedge \neg x_5 \wedge x_7 \wedge x_8 \wedge \neg x_9 \wedge x_{10} \wedge x_{11} \wedge x_{12} \wedge x_{13} \wedge x_{14} \wedge \neg x_{15}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \neg x_4 \wedge \neg x_5 \wedge \neg x_6 \wedge \neg x_7 \wedge \neg x_9 \wedge \neg x_{10} \wedge x_{11} \wedge x_{12} \wedge x_{13} \wedge x_{14} \wedge \neg x_{15}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge x_4 \wedge \neg x_5 \wedge \neg x_6 \wedge x_7 \wedge x_9 \wedge x_{10} \wedge \neg x_{11} \wedge \neg x_{12} \wedge x_{13} \wedge x_{14} \wedge \neg x_{15}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge x_4 \wedge \neg x_5 \wedge \neg x_6 \wedge x_7 \wedge x_8 \wedge x_9 \wedge x_{10} \wedge \neg x_{11} \wedge \neg x_{12} \wedge x_{13} \wedge x_{14} \wedge \neg x_{15}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge x_3 \wedge x_4 \wedge \neg x_5 \wedge \neg x_6 \wedge x_7 \wedge x_8 \wedge \neg x_9 \wedge x_{10} \wedge \neg x_{11} \wedge x_{12} \wedge x_{14} \wedge x_{16}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge x_4 \wedge \neg x_5 \wedge \neg x_6 \wedge x_7 \wedge \neg x_8 \wedge \neg x_9 \wedge x_{10} \wedge \neg x_{11} \wedge x_{12} \wedge x_{14} \wedge x_{16}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \neg x_5 \wedge x_6 \wedge x_7 \wedge x_8 \wedge x_9 \wedge \neg x_{10} \wedge x_{11} \wedge x_{12} \wedge x_{13} \wedge x_{14} \wedge x_{16}) \vee$
$(\neg x_1 \wedge x_3 \wedge x_4 \wedge \neg x_5 \wedge x_6 \wedge x_7 \wedge x_8 \wedge x_9 \wedge \neg x_{10} \wedge x_{11} \wedge x_{12} \wedge x_{13} \wedge x_{14} \wedge x_{16}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \neg x_5 \wedge \neg x_6 \wedge x_7 \wedge x_8 \wedge x_9 \wedge x_{10} \wedge \neg x_{11} \wedge x_{12} \wedge x_{13} \wedge x_{14} \wedge x_{16}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge x_3 \wedge x_4 \wedge \neg x_5 \wedge x_6 \wedge \neg x_7 \wedge x_8 \wedge \neg x_9 \wedge x_{10} \wedge x_{11} \wedge x_{13} \wedge \neg x_{14} \wedge x_{16}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \neg x_5 \wedge \neg x_6 \wedge x_7 \wedge x_8 \wedge \neg x_9 \wedge x_{10} \wedge \neg x_{11} \wedge x_{12} \wedge x_{13} \wedge \neg x_{14} \wedge x_{16}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \neg x_5 \wedge \neg x_6 \wedge \neg x_7 \wedge x_8 \wedge \neg x_9 \wedge \neg x_{10} \wedge \neg x_{11} \wedge x_{12} \wedge \neg x_{13} \wedge \neg x_{14} \wedge x_{16}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \neg x_4 \wedge \neg x_5 \wedge \neg x_6 \wedge \neg x_7 \wedge \neg x_9 \wedge x_{10} \wedge x_{11} \wedge x_{12} \wedge x_{13} \wedge x_{15} \wedge x_{16}) \vee$
$(\neg x_1 \wedge x_3 \wedge x_4 \wedge \neg x_5 \wedge x_6 \wedge x_7 \wedge x_8 \wedge \neg x_9 \wedge x_{10} \wedge x_{11} \wedge x_{12} \wedge x_{13} \wedge x_{15} \wedge x_{16}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge x_4 \wedge \neg x_5 \wedge \neg x_6 \wedge x_7 \wedge \neg x_9 \wedge \neg x_{10} \wedge x_{11} \wedge x_{12} \wedge \neg x_{13} \wedge x_{15} \wedge x_{16}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge x_4 \wedge \neg x_5 \wedge \neg x_6 \wedge x_7 \wedge x_8 \wedge \neg x_9 \wedge \neg x_{10} \wedge x_{11} \wedge x_{12} \wedge \neg x_{13} \wedge x_{15} \wedge x_{16}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge x_3 \wedge x_4 \wedge \neg x_5 \wedge x_6 \wedge \neg x_7 \wedge x_8 \wedge \neg x_9 \wedge x_{10} \wedge x_{11} \wedge x_{14} \wedge x_{15} \wedge x_{16}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge x_4 \wedge \neg x_5 \wedge \neg x_6 \wedge x_7 \wedge \neg x_8 \wedge \neg x_9 \wedge x_{10} \wedge \neg x_{11} \wedge x_{14} \wedge x_{15} \wedge x_{16}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \neg x_4 \wedge \neg x_5 \wedge \neg x_6 \wedge \neg x_7 \wedge \neg x_8 \wedge \neg x_9 \wedge \neg x_{10} \wedge \neg x_{11} \wedge x_{14} \wedge x_{15} \wedge x_{16}) \vee$
$(\neg x_1 \wedge x_3 \wedge x_4 \wedge \neg x_5 \wedge x_6 \wedge x_7 \wedge x_8 \wedge \neg x_9 \wedge x_{10} \wedge x_{11} \wedge x_{12} \wedge x_{14} \wedge x_{15} \wedge x_{16}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge x_3 \wedge x_4 \wedge x_6 \wedge \neg x_7 \wedge x_8 \wedge x_9 \wedge x_{10} \wedge x_{11} \wedge x_{13} \wedge x_{14} \wedge x_{15} \wedge x_{16}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \neg x_5 \wedge \neg x_6 \wedge x_7 \wedge x_8 \wedge x_9 \wedge x_{10} \wedge \neg x_{11} \wedge x_{13} \wedge x_{14} \wedge x_{15} \wedge x_{16}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \neg x_4 \wedge \neg x_5 \wedge \neg x_6 \wedge \neg x_7 \wedge x_{10} \wedge x_{11} \wedge x_{12} \wedge x_{13} \wedge x_{14} \wedge x_{15} \wedge x_{16}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge x_6 \wedge x_7 \wedge x_8 \wedge x_9 \wedge x_{10} \wedge x_{11} \wedge x_{12} \wedge x_{13} \wedge x_{14} \wedge x_{15} \wedge x_{16}) \vee$
$(\neg x_1 \wedge x_2 \wedge x_3 \wedge x_4 \wedge x_5 \wedge \neg x_6 \wedge x_7 \wedge x_8 \wedge x_9 \wedge x_{10} \wedge \neg x_{11} \wedge x_{12} \wedge x_{13} \wedge x_{14} \wedge x_{15}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge x_3 \wedge x_4 \wedge x_5 \wedge x_6 \wedge \neg x_7 \wedge x_8 \wedge x_9 \wedge x_{10} \wedge \neg x_{11} \wedge x_{12} \wedge x_{13} \wedge x_{14} \wedge x_{15}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge x_3 \wedge x_4 \wedge \neg x_5 \wedge x_6 \wedge \neg x_7 \wedge x_8 \wedge x_9 \wedge x_{10} \wedge x_{11} \wedge \neg x_{12} \wedge x_{13} \wedge x_{14} \wedge x_{15}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge x_3 \wedge x_4 \wedge \neg x_5 \wedge x_6 \wedge x_7 \wedge x_8 \wedge x_9 \wedge \neg x_{10} \wedge x_{11} \wedge x_{12} \wedge x_{13} \wedge x_{14} \wedge \neg x_{15}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge x_3 \wedge x_4 \wedge \neg x_5 \wedge x_6 \wedge \neg x_7 \wedge x_8 \wedge x_9 \wedge \neg x_{10} \wedge x_{11} \wedge \neg x_{12} \wedge x_{13} \wedge x_{14} \wedge \neg x_{15}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge x_4 \wedge \neg x_5 \wedge \neg x_6 \wedge x_7 \wedge \neg x_8 \wedge \neg x_9 \wedge x_{10} \wedge \neg x_{11} \wedge \neg x_{12} \wedge x_{13} \wedge \neg x_{14} \wedge \neg x_{15}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \neg x_4 \wedge \neg x_5 \wedge \neg x_6 \wedge \neg x_7 \wedge \neg x_8 \wedge \neg x_9 \wedge \neg x_{10} \wedge \neg x_{11} \wedge \neg x_{12} \wedge \neg x_{13} \wedge \neg x_{14} \wedge \neg x_{15}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge x_3 \wedge x_4 \wedge x_5 \wedge x_6 \wedge \neg x_7 \wedge x_8 \wedge x_9 \wedge x_{10} \wedge \neg x_{11} \wedge x_{12} \wedge x_{13} \wedge x_{14} \wedge x_{16}) \vee$
$(\neg x_1 \wedge x_2 \wedge x_3 \wedge x_4 \wedge \neg x_5 \wedge x_6 \wedge x_7 \wedge x_8 \wedge x_9 \wedge \neg x_{10} \wedge x_{11} \wedge x_{12} \wedge x_{13} \wedge x_{15} \wedge x_{16}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge x_3 \wedge x_4 \wedge \neg x_5 \wedge x_6 \wedge \neg x_7 \wedge x_8 \wedge x_9 \wedge \neg x_{10} \wedge x_{11} \wedge x_{12} \wedge x_{13} \wedge x_{15} \wedge x_{16}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge x_4 \wedge \neg x_5 \wedge x_6 \wedge x_7 \wedge \neg x_8 \wedge x_9 \wedge \neg x_{10} \wedge x_{11} \wedge x_{12} \wedge x_{13} \wedge x_{15} \wedge x_{16}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge x_3 \wedge x_4 \wedge x_5 \wedge x_6 \wedge \neg x_7 \wedge x_9 \wedge x_{10} \wedge x_{11} \wedge x_{12} \wedge x_{13} \wedge x_{14} \wedge x_{15} \wedge x_{16}) \vee$
$(x_2 \wedge x_3 \wedge x_4 \wedge x_5 \wedge x_6 \wedge x_7 \wedge x_8 \wedge x_9 \wedge x_{10} \wedge x_{11} \wedge x_{12} \wedge x_{13} \wedge x_{14} \wedge x_{15} \wedge x_{16}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \neg x_5 \wedge \neg x_6 \wedge \neg x_7 \wedge x_8 \wedge \neg x_9 \wedge x_{11} \wedge x_{14} \wedge x_{15} \wedge x_{16}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge x_4 \wedge \neg x_5 \wedge \neg x_6 \wedge \neg x_7 \wedge x_8 \wedge \neg x_9 \wedge \neg x_{10} \wedge x_{11} \wedge x_{13} \wedge x_{14}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \neg x_5 \wedge \neg x_6 \wedge \neg x_7 \wedge x_8 \wedge \neg x_9 \wedge x_{11} \wedge x_{12} \wedge x_{13} \wedge x_{14} \wedge x_{15}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge x_4 \wedge \neg x_5 \wedge \neg x_6 \wedge x_7 \wedge x_9 \wedge x_{10} \wedge \neg x_{11} \wedge x_{13} \wedge x_{14} \wedge x_{16}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge x_4 \wedge \neg x_5 \wedge x_7 \wedge \neg x_9 \wedge x_{10} \wedge x_{11} \wedge x_{12} \wedge x_{13} \wedge x_{15} \wedge x_{16}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \neg x_5 \wedge \neg x_6 \wedge x_8 \wedge \neg x_9 \wedge x_{10} \wedge x_{11} \wedge x_{12} \wedge x_{13} \wedge x_{15} \wedge x_{16}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge x_4 \wedge \neg x_5 \wedge x_7 \wedge x_8 \wedge \neg x_9 \wedge x_{10} \wedge x_{11} \wedge x_{12} \wedge x_{13} \wedge x_{15} \wedge x_{16}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge x_4 \wedge \neg x_5 \wedge \neg x_6 \wedge x_7 \wedge \neg x_9 \wedge x_{11} \wedge x_{12} \wedge x_{14} \wedge x_{15} \wedge x_{16}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge x_4 \wedge \neg x_5 \wedge x_7 \wedge \neg x_9 \wedge x_{10} \wedge x_{11} \wedge x_{12} \wedge x_{14} \wedge x_{15} \wedge x_{16}) \vee$

Table A.14 – Continued on next page

Table A.14 – continued from previous page

$(\neg x_1 \wedge \neg x_2 \wedge x_4 \wedge \neg x_5 \wedge x_7 \wedge x_8 \wedge \neg x_9 \wedge x_{10} \wedge x_{11} \wedge x_{12} \wedge x_{14} \wedge x_{15} \wedge x_{16}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge x_4 \wedge \neg x_5 \wedge x_7 \wedge x_{10} \wedge x_{11} \wedge x_{12} \wedge x_{13} \wedge x_{14} \wedge x_{15} \wedge x_{16}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \neg x_5 \wedge \neg x_6 \wedge x_8 \wedge x_{10} \wedge x_{11} \wedge x_{12} \wedge x_{13} \wedge x_{14} \wedge x_{15} \wedge x_{16}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge x_4 \wedge \neg x_5 \wedge x_7 \wedge x_8 \wedge x_{10} \wedge x_{11} \wedge x_{12} \wedge x_{13} \wedge x_{14} \wedge x_{15} \wedge x_{16}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge x_4 \wedge \neg x_5 \wedge x_7 \wedge x_8 \wedge \neg x_9 \wedge x_{10} \wedge x_{11} \wedge x_{12} \wedge x_{13} \wedge x_{14} \wedge x_{15}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge x_4 \wedge \neg x_5 \wedge \neg x_6 \wedge x_7 \wedge x_8 \wedge x_9 \wedge x_{10} \wedge \neg x_{12} \wedge x_{13} \wedge x_{14} \wedge x_{15}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge x_4 \wedge \neg x_5 \wedge x_7 \wedge x_8 \wedge x_9 \wedge x_{10} \wedge x_{11} \wedge \neg x_{12} \wedge x_{13} \wedge x_{14} \wedge x_{15}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge x_3 \wedge x_4 \wedge \neg x_5 \wedge \neg x_6 \wedge x_7 \wedge x_8 \wedge \neg x_9 \wedge x_{10} \wedge x_{12} \wedge x_{13} \wedge x_{14} \wedge \neg x_{15}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge x_4 \wedge \neg x_5 \wedge \neg x_6 \wedge x_7 \wedge \neg x_8 \wedge \neg x_9 \wedge x_{10} \wedge x_{12} \wedge x_{13} \wedge x_{14} \wedge \neg x_{15}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge x_3 \wedge x_4 \wedge \neg x_5 \wedge x_6 \wedge \neg x_7 \wedge x_8 \wedge x_9 \wedge \neg x_{10} \wedge x_{11} \wedge x_{13} \wedge x_{14} \wedge x_{16}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge x_4 \wedge \neg x_5 \wedge x_6 \wedge x_7 \wedge x_9 \wedge \neg x_{10} \wedge x_{11} \wedge x_{12} \wedge x_{13} \wedge x_{14} \wedge x_{16}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge x_4 \wedge \neg x_5 \wedge \neg x_6 \wedge x_7 \wedge \neg x_8 \wedge \neg x_9 \wedge x_{10} \wedge \neg x_{11} \wedge x_{13} \wedge \neg x_{14} \wedge x_{16}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge x_4 \wedge \neg x_5 \wedge \neg x_6 \wedge x_7 \wedge x_8 \wedge \neg x_9 \wedge x_{10} \wedge \neg x_{11} \wedge x_{12} \wedge x_{13} \wedge \neg x_{14} \wedge x_{16}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \neg x_4 \wedge \neg x_5 \wedge \neg x_6 \wedge \neg x_7 \wedge \neg x_8 \wedge \neg x_9 \wedge \neg x_{10} \wedge \neg x_{11} \wedge \neg x_{13} \wedge \neg x_{14} \wedge x_{16}) \vee$
$(\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge x_4 \wedge x_6 \wedge x_7 \wedge x_9 \wedge x_{10} \wedge x_{11} \wedge x_{12} \wedge x_{13} \wedge x_{14} \wedge x_{15} \wedge x_{16}) \vee$
$(\neg x_1 \wedge x_3 \wedge x_4 \wedge x_6 \wedge x_7 \wedge x_8 \wedge x_9 \wedge x_{10} \wedge x_{11} \wedge x_{12} \wedge x_{13} \wedge x_{14} \wedge x_{15} \wedge x_{16}) \vee$
$(\neg x_1 \wedge x_2 \wedge x_3 \wedge x_4 \wedge x_5 \wedge x_7 \wedge x_8 \wedge x_9 \wedge x_{10} \wedge x_{11} \wedge x_{12} \wedge x_{13} \wedge x_{14} \wedge x_{15} \wedge x_{16})$

Table A.15: Minimized CNF of lex-constraints for $M_{4\times 4}$, domain 2

$(\neg x_1 \vee x_2) \wedge$
$(\neg x_2 \vee x_3) \wedge$
$(\neg x_3 \vee x_4) \wedge$
$(\neg x_1 \vee x_5) \wedge$
$(\neg x_1 \vee x_6) \wedge$
$(\neg x_2 \vee x_7) \wedge$
$(\neg x_5 \vee x_9) \wedge$
$(\neg x_5 \vee x_{10}) \wedge$
$(\neg x_9 \vee x_{13}) \wedge$
$(x_2 \vee \neg x_5 \vee x_6) \wedge$
$(\neg x_2 \vee x_5 \vee x_6) \wedge$
$(x_3 \vee \neg x_6 \vee x_7) \wedge$
$(\neg x_3 \vee x_6 \vee x_7) \wedge$
$(\neg x_3 \vee \neg x_7 \vee x_8) \wedge$
$(x_4 \vee \neg x_7 \vee x_8) \wedge$
$(x_6 \vee \neg x_9 \vee x_{10}) \wedge$
$(\neg x_6 \vee x_9 \vee x_{10}) \wedge$
$(x_3 \vee \neg x_6 \vee x_{11}) \wedge$
$(\neg x_6 \vee x_9 \vee x_{11}) \wedge$
$(\neg x_7 \vee x_{10} \vee x_{11}) \wedge$
$(\neg x_9 \vee \neg x_{10} \vee x_{14}) \wedge$
$(\neg x_{10} \vee x_{13} \vee x_{14}) \wedge$
$(x_5 \vee x_7 \vee \neg x_{10} \vee x_{11}) \wedge$
$(\neg x_3 \vee \neg x_7 \vee \neg x_{11} \vee x_{12}) \wedge$
$(x_4 \vee \neg x_7 \vee \neg x_{11} \vee x_{12}) \wedge$
$(x_6 \vee x_8 \vee \neg x_{11} \vee x_{12}) \wedge$
$(\neg x_8 \vee x_9 \vee x_{11} \vee x_{12}) \wedge$

Table A.15 – Continued on next page

Table A.15 – continued from previous page

$(\neg x_9 \vee \neg x_{10} \vee \neg x_{11} \vee x_{15}) \wedge$
$(\neg x_{10} \vee \neg x_{11} \vee x_{13} \vee x_{15}) \wedge$
$(\neg x_{12} \vee x_{14} \vee x_{15} \vee x_{16}) \wedge$
$(\neg x_{10} \vee \neg x_{11} \vee \neg x_{12} \vee x_{13} \vee x_{16}) \wedge$
$(x_4 \vee \neg x_7 \vee \neg x_{11} \vee \neg x_{15} \vee x_{16}) \wedge$
$(x_6 \vee x_8 \vee \neg x_{11} \vee \neg x_{15} \vee x_{16}) \wedge$
$(x_4 \vee x_{10} \vee x_{12} \vee \neg x_{15} \vee x_{16}) \wedge$
$(x_3 \vee \neg x_8 \vee x_9 \vee x_{11} \vee \neg x_{14} \vee x_{15}) \wedge$
$(\neg x_2 \vee \neg x_6 \vee x_{11}) \wedge$
$(\neg x_2 \vee \neg x_{10} \vee x_{15}) \wedge$
$(\neg x_2 \vee \neg x_{11} \vee x_{16}) \wedge$
$(\neg x_4 \vee x_5 \vee x_7 \vee x_8) \wedge$
$(\neg x_5 \vee \neg x_6 \vee \neg x_7 \vee x_{11}) \wedge$
$(x_7 \vee x_8 \vee \neg x_{10} \vee x_{11}) \wedge$
$(\neg x_5 \vee \neg x_7 \vee \neg x_8 \vee x_{12}) \wedge$
$(\neg x_7 \vee \neg x_8 \vee x_{10} \vee x_{12}) \wedge$
$(x_7 \vee x_8 \vee \neg x_{11} \vee x_{12}) \wedge$
$(x_7 \vee \neg x_9 \vee x_{11} \vee x_{12}) \wedge$
$(x_9 \vee x_{10} \vee \neg x_{13} \vee x_{14}) \wedge$
$(x_{10} \vee x_{11} \vee \neg x_{13} \vee x_{14}) \wedge$
$(x_{10} \vee x_{12} \vee \neg x_{13} \vee x_{14}) \wedge$
$(x_3 \vee \neg x_6 \vee \neg x_{10} \vee x_{15}) \wedge$
$(x_6 \vee \neg x_{11} \vee x_{14} \vee x_{15}) \wedge$
$(\neg x_7 \vee \neg x_{11} \vee x_{14} \vee x_{15}) \wedge$
$(x_{10} \vee \neg x_{11} \vee x_{14} \vee x_{15}) \wedge$
$(\neg x_5 \vee \neg x_8 \vee \neg x_{11} \vee x_{16}) \wedge$
$(\neg x_6 \vee \neg x_7 \vee \neg x_8 \vee x_9 \vee x_{12}) \wedge$
$(x_7 \vee \neg x_{10} \vee \neg x_{11} \vee \neg x_{14} \vee x_{15}) \wedge$
$(x_5 \vee x_7 \vee x_{11} \vee \neg x_{14} \vee x_{15}) \wedge$
$(x_4 \vee x_{11} \vee x_{12} \vee \neg x_{14} \vee x_{15}) \wedge$
$(x_9 \vee x_{11} \vee x_{12} \vee \neg x_{14} \vee x_{15}) \wedge$
$(\neg x_9 \vee \neg x_{10} \vee \neg x_{11} \vee \neg x_{12} \vee x_{16}) \wedge$
$(\neg x_6 \vee \neg x_8 \vee x_9 \vee x_{12} \vee x_{16}) \wedge$
$(x_{10} \vee \neg x_{11} \vee \neg x_{12} \vee x_{14} \vee x_{16}) \wedge$
$(\neg x_3 \vee \neg x_7 \vee \neg x_{11} \vee \neg x_{15} \vee x_{16}) \wedge$
$(\neg x_3 \vee x_{11} \vee x_{12} \vee \neg x_{15} \vee x_{16}) \wedge$
$(x_4 \vee x_{11} \vee x_{12} \vee \neg x_{15} \vee x_{16}) \wedge$
$(x_8 \vee x_{11} \vee x_{12} \vee \neg x_{15} \vee x_{16}) \wedge$
$(\neg x_6 \vee x_7 \vee \neg x_{12} \vee x_{15} \vee x_{16}) \wedge$
$(x_3 \vee \neg x_9 \vee \neg x_{12} \vee x_{15} \vee x_{16}) \wedge$
$(\neg x_9 \vee x_{11} \vee \neg x_{12} \vee x_{15} \vee x_{16}) \wedge$
$(\neg x_6 \vee \neg x_{10} \vee x_{12} \vee x_{15} \vee x_{16}) \wedge$
$(x_7 \vee \neg x_{12} \vee x_{13} \vee x_{15} \vee x_{16}) \wedge$
$(\neg x_8 \vee \neg x_{12} \vee x_{13} \vee x_{15} \vee x_{16}) \wedge$
$(x_{11} \vee x_{13} \vee \neg x_{14} \vee x_{15} \vee x_{16}) \wedge$
$(x_2 \vee \neg x_7 \vee \neg x_8 \vee x_{10} \vee \neg x_{13} \vee x_{14}) \wedge$
$(x_3 \vee \neg x_8 \vee \neg x_{11} \vee \neg x_{12} \vee x_{14} \vee x_{16}) \wedge$

Table A.15 – Continued on next page

Table A.15 – continued from previous page

$(\neg x_8 \vee \neg x_9 \vee x_{10} \vee x_{12} \vee \neg x_{15} \vee x_{16})$

Table A.16: Comparision of Constraints, $M_{4\times4}$, domain 2

	Nr of const	Nr of sol	Nr of lit	Speed	Speed
Espresso - exact CNF	75	317	297	45.50	133.50
Espresso - exact DNF	105	317	1435	229.9	904.6
Lex, entire symmetry group	576	317	129024	533.30	1050.90
Lex, after simplifications	270	317	30926	118.60	257.50
Lex, approximated min set[a]	14	364	1204	5.50	26.20
Lex, row-col	6	650	336	2.40	28.5

[a] The largest n used in the approximated minimal set is 2

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