# Implied Constraints for CP solvers

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# Background: Combinatorial optimisation

**Combinatorial optimisation** consists of finding an object from a finite set of objects:

- The set of feasible solutions is discrete or can be discretised.
- The goal is to find a solution, or all solutions, or a best solution.
- Examples:
  - the travelling tournament problem.
  - the nurse scheduling problem.

**Constraint programming** (CP) is a set of techniques and tools for effectively modelling and efficiently solving hard combinatorial problems.

CP solving = inference + search

# Background: CP Modelling

**Constraints** form the vocabulary of a CP modelling language: they allow a modeller to express commonly occurring substructures.

**Example**

The $\textsc{Distinct}(x, y, z)$ constraint, over the variables $x$,$y$, and $z$ with domains $x \in \{1, 2\}, y \in \{1, 2\}$, and $z \in \{2, 3, 4, 5\}$

A **constraint problem** is a conjunction of constraints.

**Example**

$$\textsc{Distinct}(x, y, z) \wedge x + y < z$$

# Background: CP Inference and Search

A constraint comes with a **propagator**, which removes impossible values from the domains of its variables.

## Example

$\textsc{Distinct}(x, y, z)$ with $x = \{1, 2\}, y = \{1, 2\}$, and $z = \{\cancel{2}, 3, 4, 5\}$

After the propagators have removed the values they can, the solver will begin a systematic search if need be:

- Select a variable
- Select a value (or a range of values)
- Propagate again on the domain of the variables

## Example

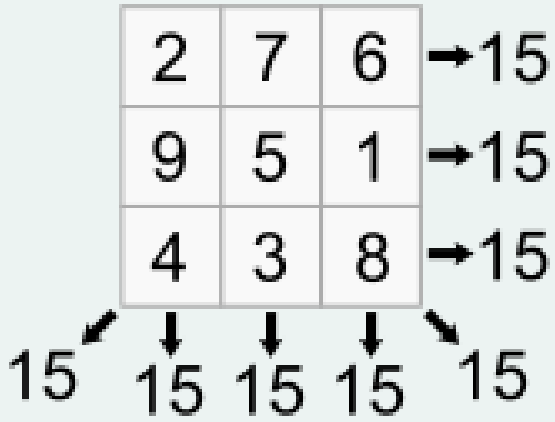
$x = 1$: $\textsc{Distinct}(x, y, z)$, $x = \{1, \cancel{2}\}, y = \{\cancel{1}, 2\}$, and $z = \{3, 4, 5\}$
$x \neq 1$: $\textsc{Distinct}(x, y, z)$, $x = \{\cancel{1}, 2\}, y = \{1, \cancel{2}\}$, and $z = \{3, 4, 5\}$

# Background: Implied Constraints

An **implied constraint** is a constraint that logically follows from other constraints.

Implied constraints may improve propagation or running time, without losing any solutions.

## Example (Magic Square)



# Background

Although modern CP solvers have many built-in constraints, often a constraint that one is looking for is not there. In such cases, the choices are:

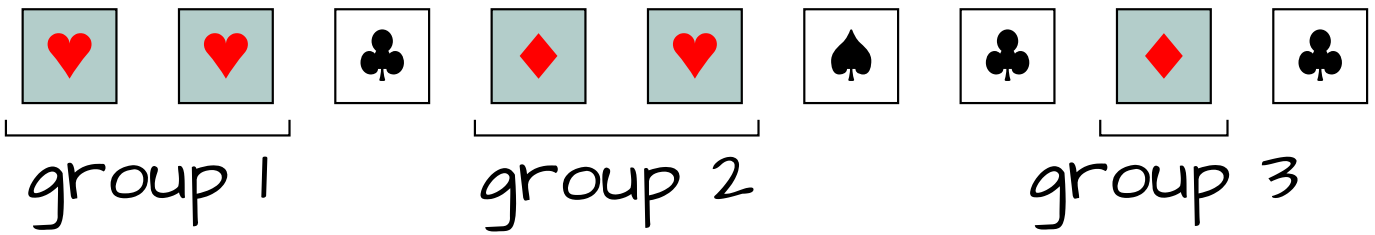
- to reformulate the model without the needed constraint.
- to write one's own propagator.

For example, deterministic finite automata (DFA) augmented with accumulators can encode a constraint on a sequence $S$ of variables [BCP04].

# Example: The Group Constraint

The $\textsc{Group}(S, N, W)$ constraint holds if and only if there are $N$ contiguous subsequences of values from the given set $W$ in the sequence $S$ of variables.

$$\textsc{Group}([\heartsuit, \heartsuit, \clubsuit, \diamondsuit, \heartsuit, \spadesuit, \clubsuit, \diamondsuit, \clubsuit], 3, \{\heartsuit, \diamondsuit\})$$



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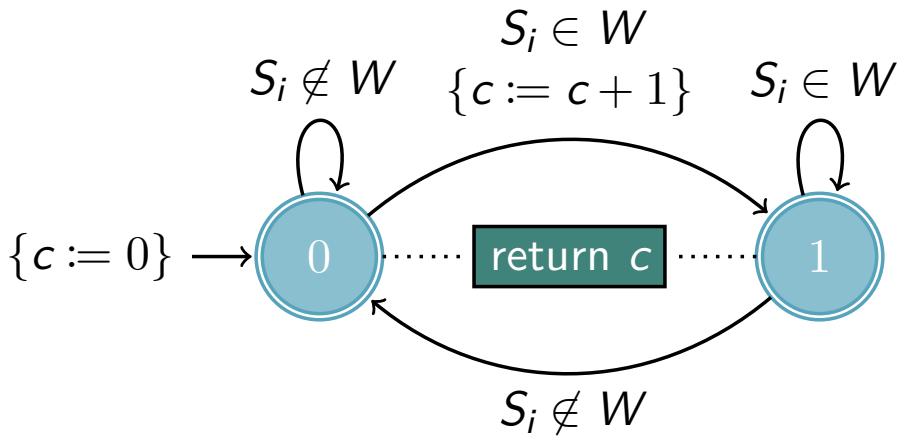
| $i$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| $c_i$ | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 3 | 3 |



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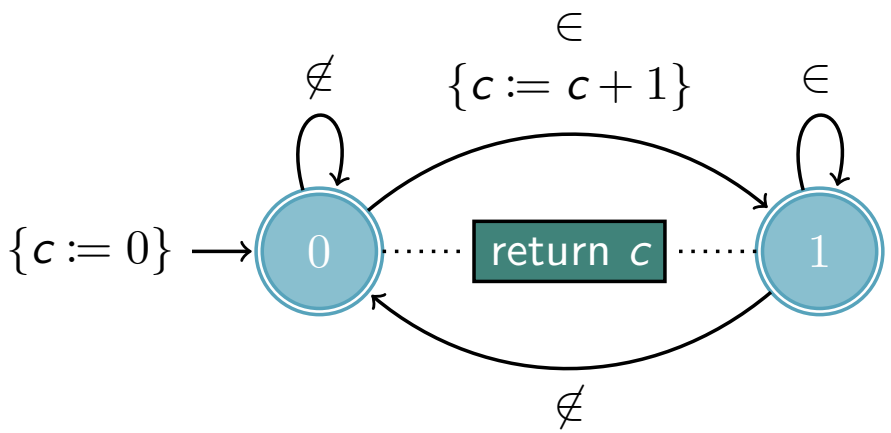


# Motivation

In general, it is hard to do perfect propagation efficiently for constraints encoded via automata [BCP04].

**Implied constraints on accumulators** are a way to improve propagation, as these may trigger extra propagation at each node of the search tree [BCRT05, FFP13].

# Our Objective

Generate implied constraints that improve propagation for constraints encoded via automata with at least one accumulator.

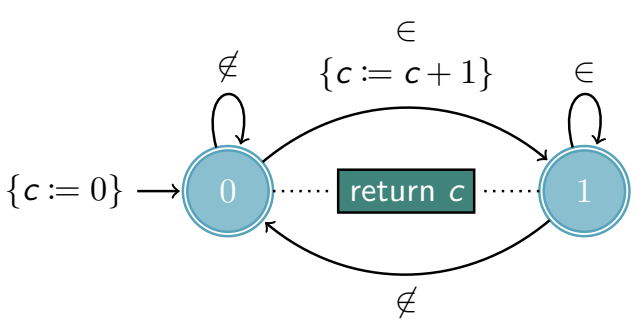
- The generation is specific to the given automaton $\mathcal{A}$
- The implied constraints are generated offline.
- The implied constraints are of the form:

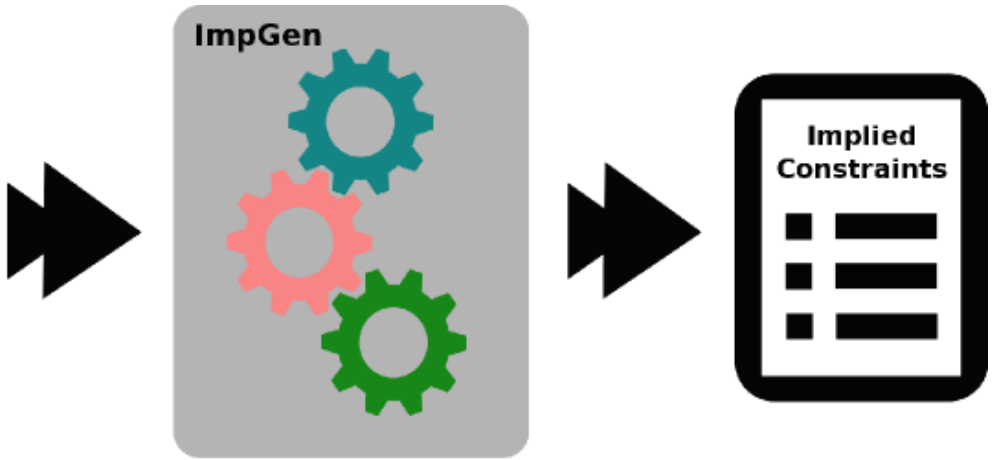
$$\alpha_1 y_1 + \cdots + \alpha_k y_k + \beta \geq 0$$

where the $y_i$ are the accumulators of $\mathcal{A}$ and the weights $\alpha_i$ and $\beta$ are to be found.

# Our Work

We developed a tool based on Farkas' Lemma.




Options

# Farkas' Lemma

A set of $e$ linear inequalities over real-valued variables $y_i$ has another linear inequality over the same variables as a logical consequence if the latter is equal to a linear combination of the former.

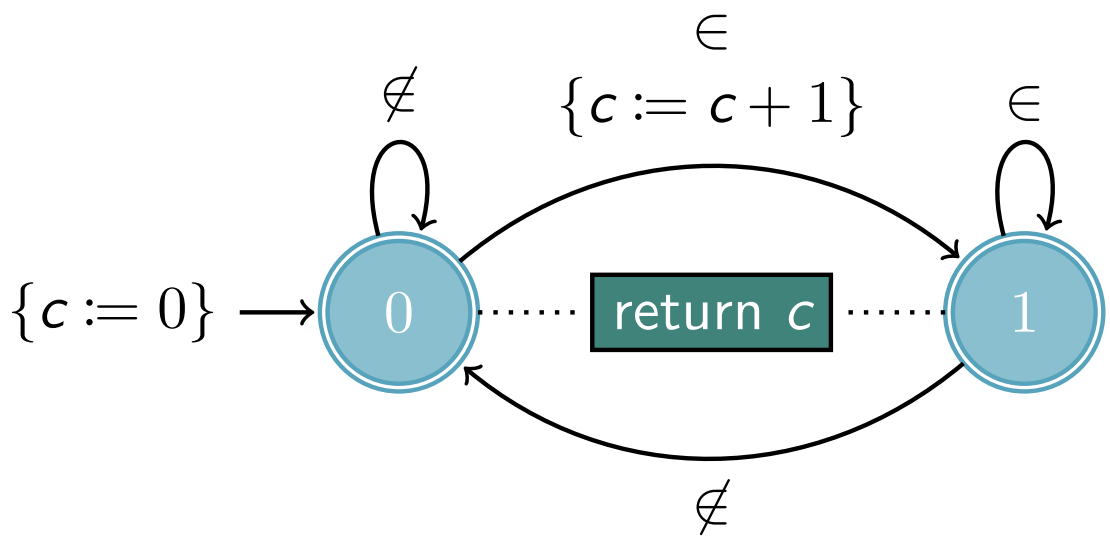
$$\begin{array}{c|c} \lambda_1 & a_{11}y_1 + \cdots + a_{1k}y_k + b_1 \geq 0 \\ \vdots & \vdots \quad \vdots \quad \vdots \quad \vdots \\ \lambda_e & a_{e1}y_1 + \cdots + a_{ek}y_k + b_e \geq 0 \\ \hline & \alpha_1 y_1 + \cdots + \alpha_k y_k + \beta \geq 0 \end{array}$$

That is, we need to find values to the $\alpha_j$, $\beta$, and $\lambda_i$ such that:

- $\alpha_j = \sum_{i=1}^{e} \lambda_i a_{ij}$ for $1 \leq j \leq k$
- $\beta \geq \sum_{i=1}^{e} \lambda_i b_i$
- $\lambda_i \geq 0$, except if the $i$-th linear constraint is an equality

# Using Farkas' Lemma (example)

Consider implied constraints of the template $\alpha c + \beta \geq 0$ and the GROUP DFA:

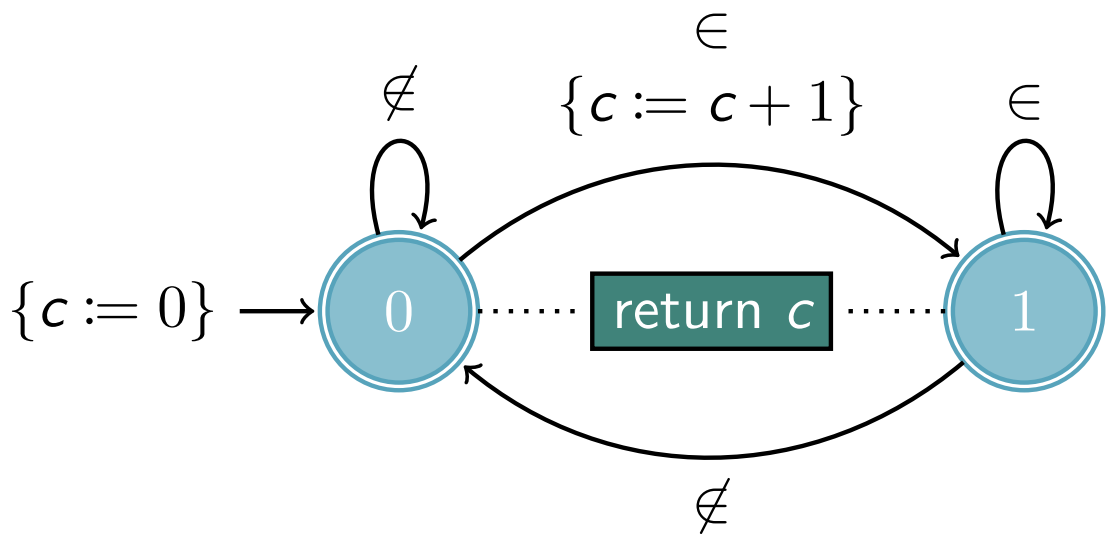


# Using Farkas' Lemma (example)

Initialisation:

| $\lambda_0$ | $c \qquad = 0$ |
|---|---|
| | $\alpha c + \beta \geq 0$ |

$$\lambda_0 = \alpha$$
$$\beta \geq 0$$



# Using Farkas' Lemma (example)

Transition $0 \to 1$:

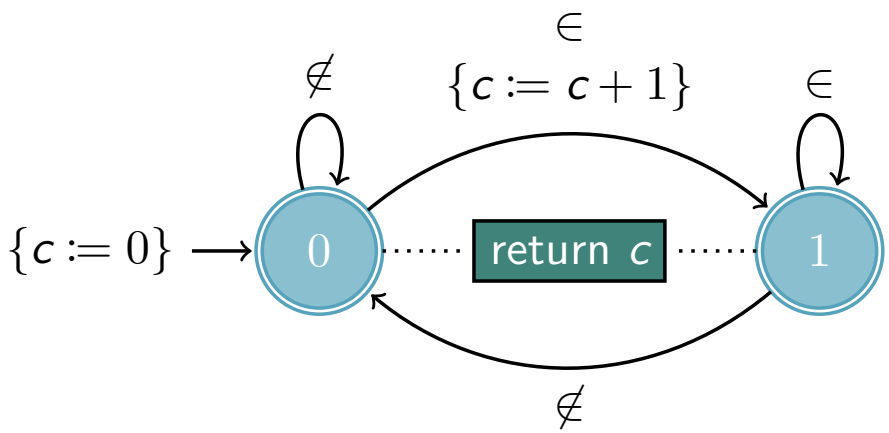
$$\frac{\lambda_1 \mid \alpha c + \beta \geq 0}{\mid \alpha(c+1) + \beta \geq 0}$$

All the other transitions:

$$\frac{\lambda_2 \mid \alpha c + \beta \geq 0}{\mid \alpha c + \beta \geq 0}$$

subject to $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$.

For example, the solution $\alpha = 1 \wedge \beta = 0$, corresponds to the implied constraint $c \geq 0$.




The implied constraints can then be added to the model in order to improve propagation.

# Our Work

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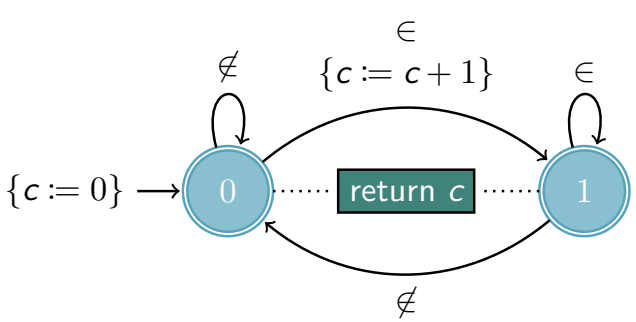
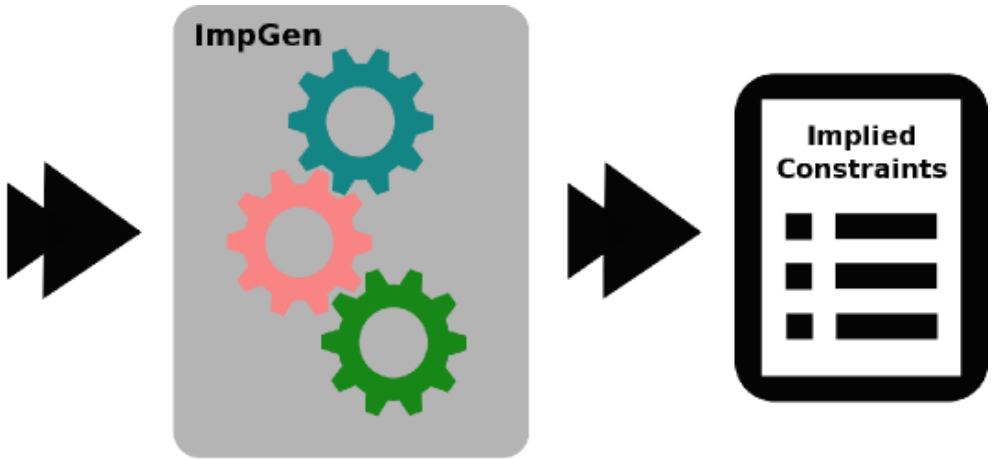
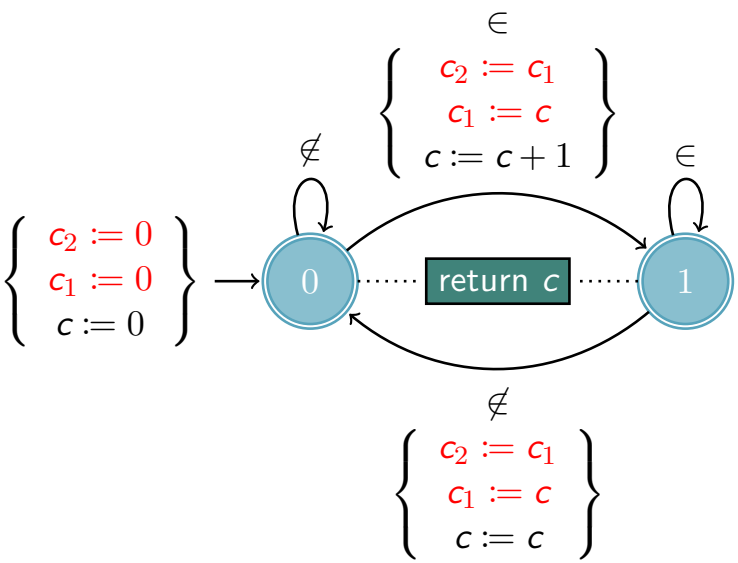



Options

# Option 1: History Accumulators

The tool can add $h$ accumulators $c_i$ to the DFA in order to store the previous $h$ values of each accumulator $c$.




Consider $h = 2$.

The template now is
$\alpha c + \alpha_1 c_1 + \alpha_2 c_2 + \beta \geq 0$.

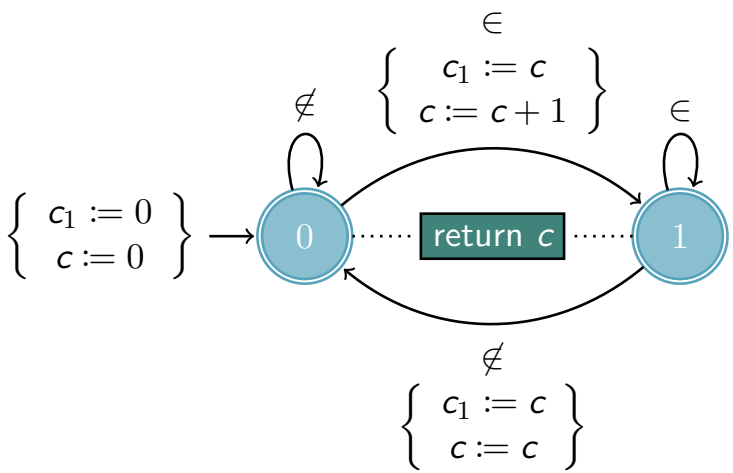
For example, we now find the implied constraints $c_1 \leq c$ and $c \leq c_2 + 1$ (the latter also requires Options 2 and 3).

# Option 2: State Variable

The tool can add a term $\rho q$ for the current state $q$ to the template $\alpha_1 y_1 + ... + \alpha_k y_k + \rho q + \beta \geq 0$.

For example, for a history of length $h \geq 1$, the tool now finds the implied constraint:

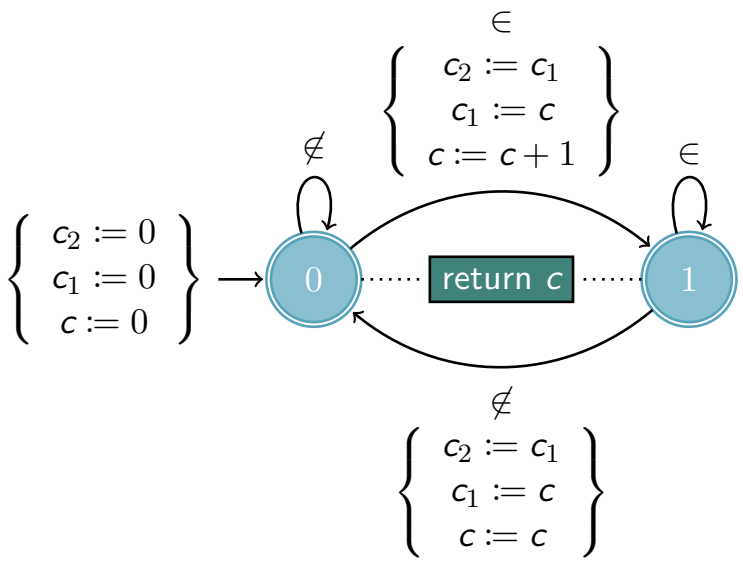
$$c - c_1 \leq q$$



# Option 3: Number of linear systems (or turns)

The tool cannot generate the previously mentioned implied constraint $c \leq c_2 + 1$ after solving the first linear system.

Generating the implied constraint $c \leq c_2 + 1$ requires the prior implied constraints $c - c_1 \leq q$ and $c_2 \leq c_1$. Inferring the latter also requires the prior implied constraint $c_1 \leq c$.



# Experiments

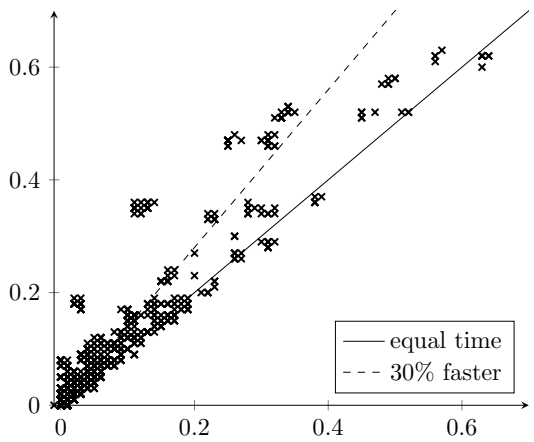
- Constraints in isolation:
  - Randomly generated instances.
  - Instances are divided into satisfiable and unsatisfiable ones.

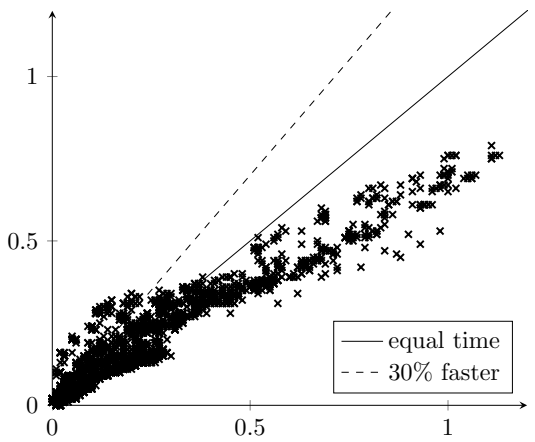
- Entire constraint problems:
  - Randomly generated instances.
  - Designed to be hard problem instances.
  - Half of the constraints in the problem instance are all of the same kind, and it corresponds to the constraint being tested.

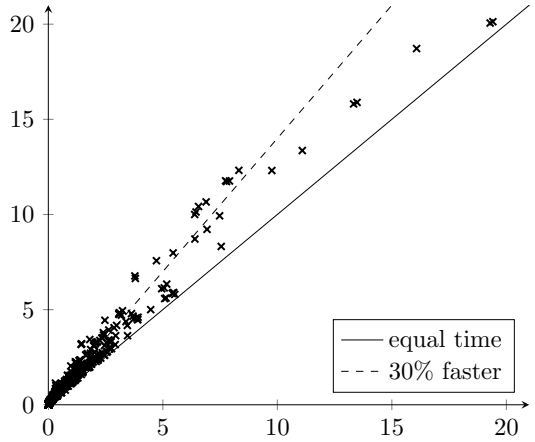
All experiments were run in SICStus Prolog 4.2.

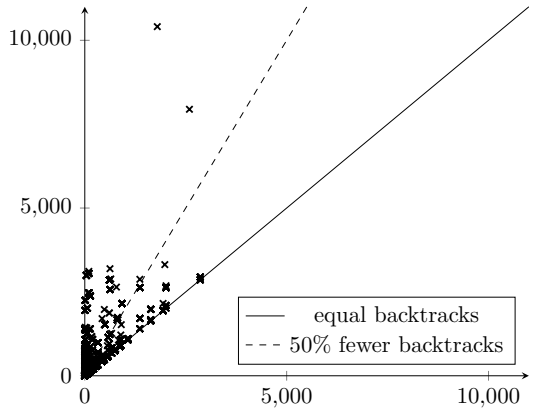
# Constraints in Isolation (satisfiable)

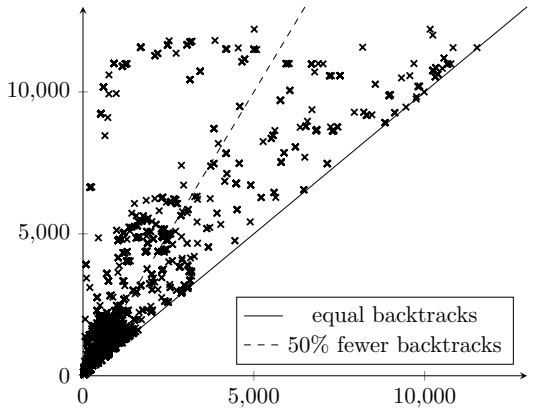
Seconds (top) and backtracks (bottom) to find all solutions to satisfiable instances of Group, FullGroupNval, and Inflexion:

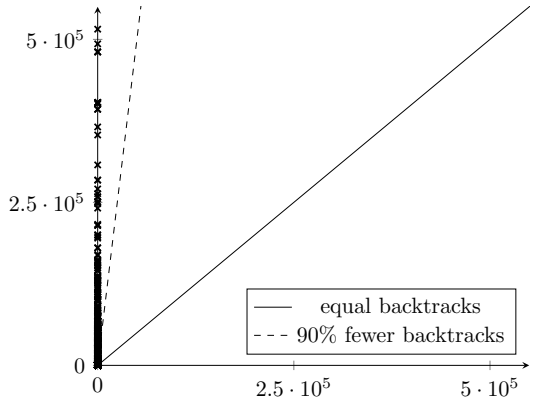


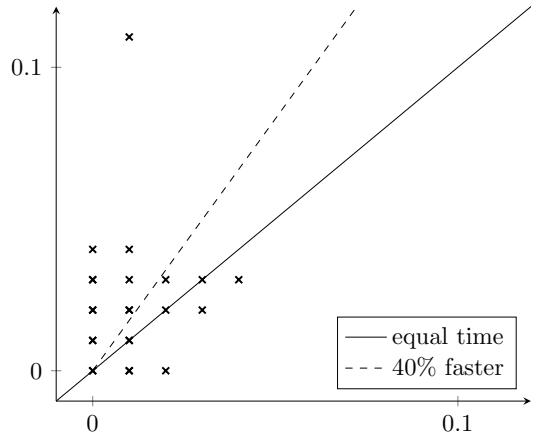



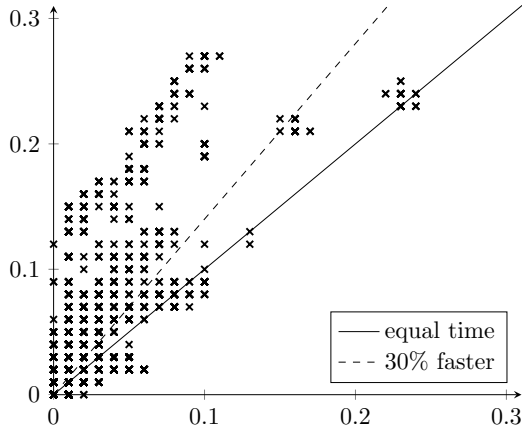



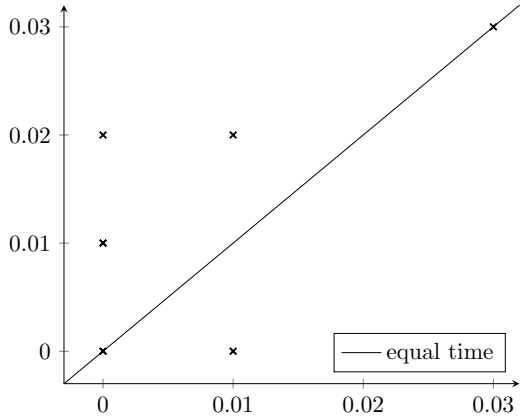



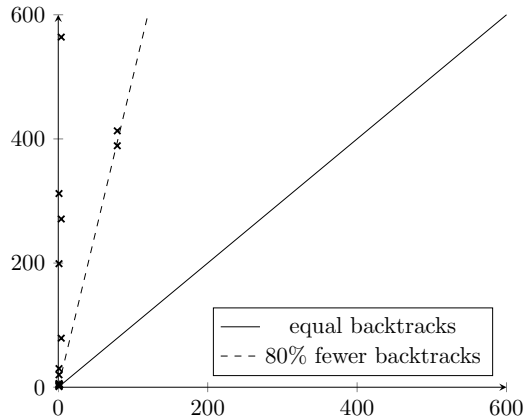
# Constraints in Isolation (unsatisfiable)

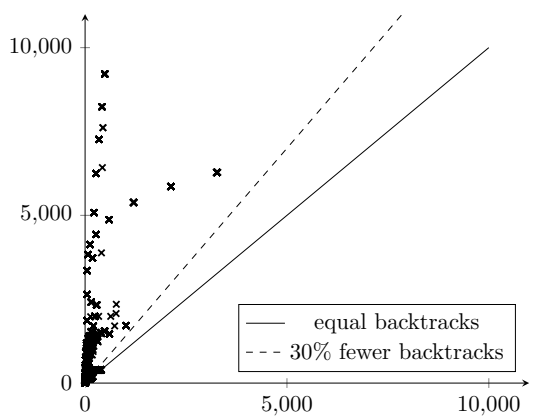
Seconds (top) and backtracks (bottom) to show unsatisfiability on instances of Group, FullGroupNval, and Inflexion:

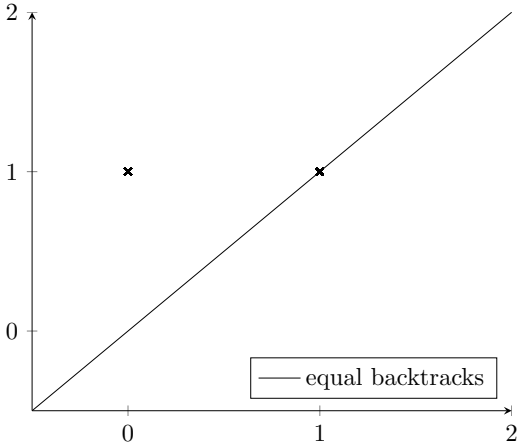


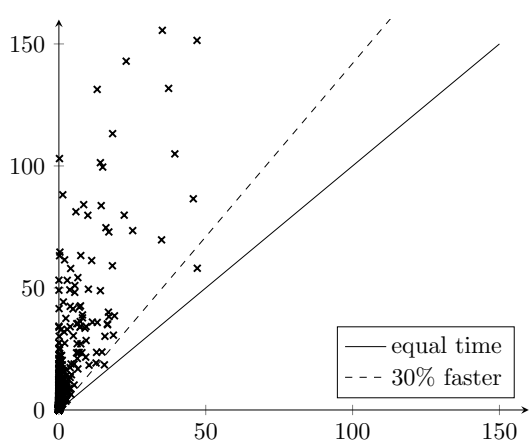



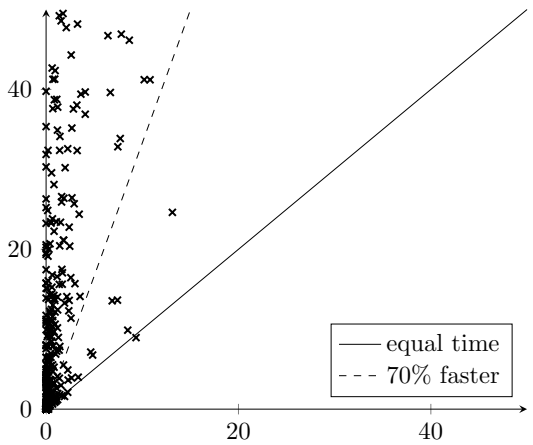



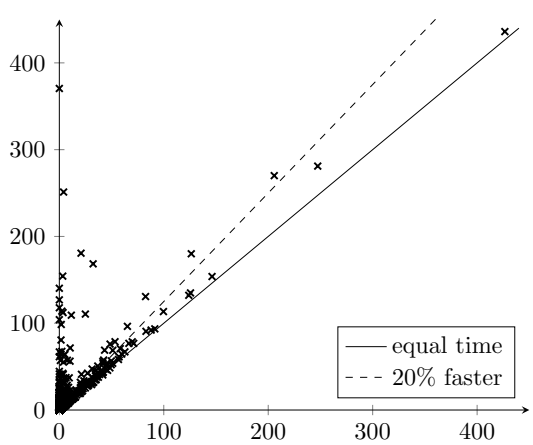



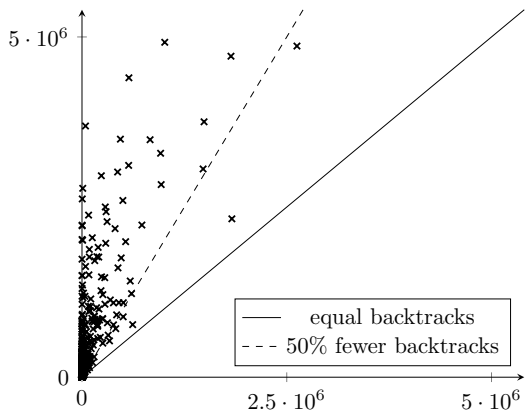
# Entire Constraint Problems

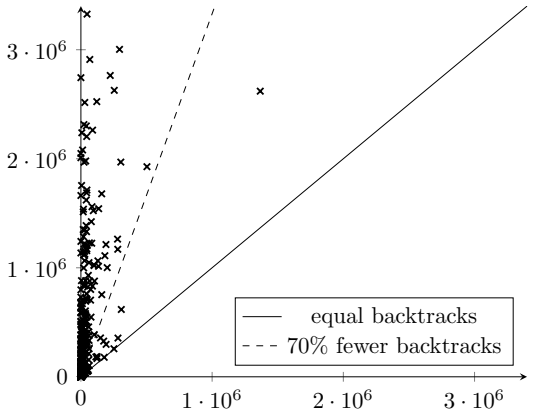
Seconds (top) and backtracks (bottom) to maximise a sum in problem instances involving Group, FullGroupNval, and Inflexion:

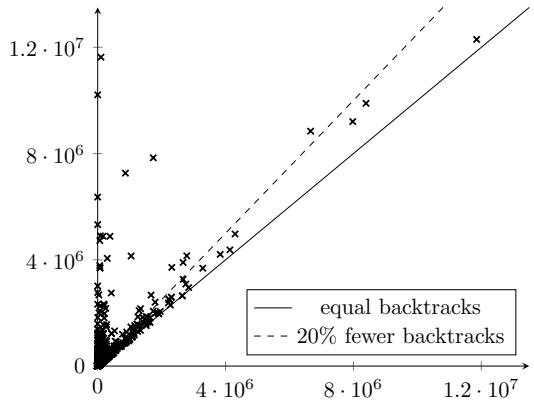


# Conclusion

- We developed a fully automated parametric tool that suggests, in an offline process, a set of linear constraints that are implied by an automaton with accumulators.
- We showed that a suitable choice, by the user, among the suggested implied constraints can considerably improve solving time.
- The generated implied constraints have also been very successfully used in the context of integer programming [Ara15].

# Related Work

- In [BCF+14], we generate implied constraints for automata where the returned value is the same whether the automaton consumes the sequence of variables or its reverse.
  - Like here, the implied constraints are on the accumulator variables and state variables
  - Unlike here, the generation is limited to the indicated particular case and is manual in most sub-cases.

- In [BCRT05], graph invariants are used to generate implied constraints.
  - Our approach does not require a database of precomputed invariants.

# Future Work

- Use a richer template for implied constraints:
  - Disjunction;
    a motivating example is in [FFP13].
  - Non-linear templates, for instance, multiplication of accumulators;
    a motivating example is in [BCRT05].

- Add other options to the tool:
  - Adding a term $\rho i$ for the index variable $i$ to the template $\alpha_1 y_1 + \cdots + \alpha_k y_k + \rho i + \beta \geq 0$.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