Symmetry-breaking for SAT: The Mysteries of Logic Minimization

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Outline

- **Motivation and Goals**
- Mathematical Background
- Previous work
- New Constructions for Single Perms
- Symmetry-breaking for Multiple Generators
- Conclusions and On-going work

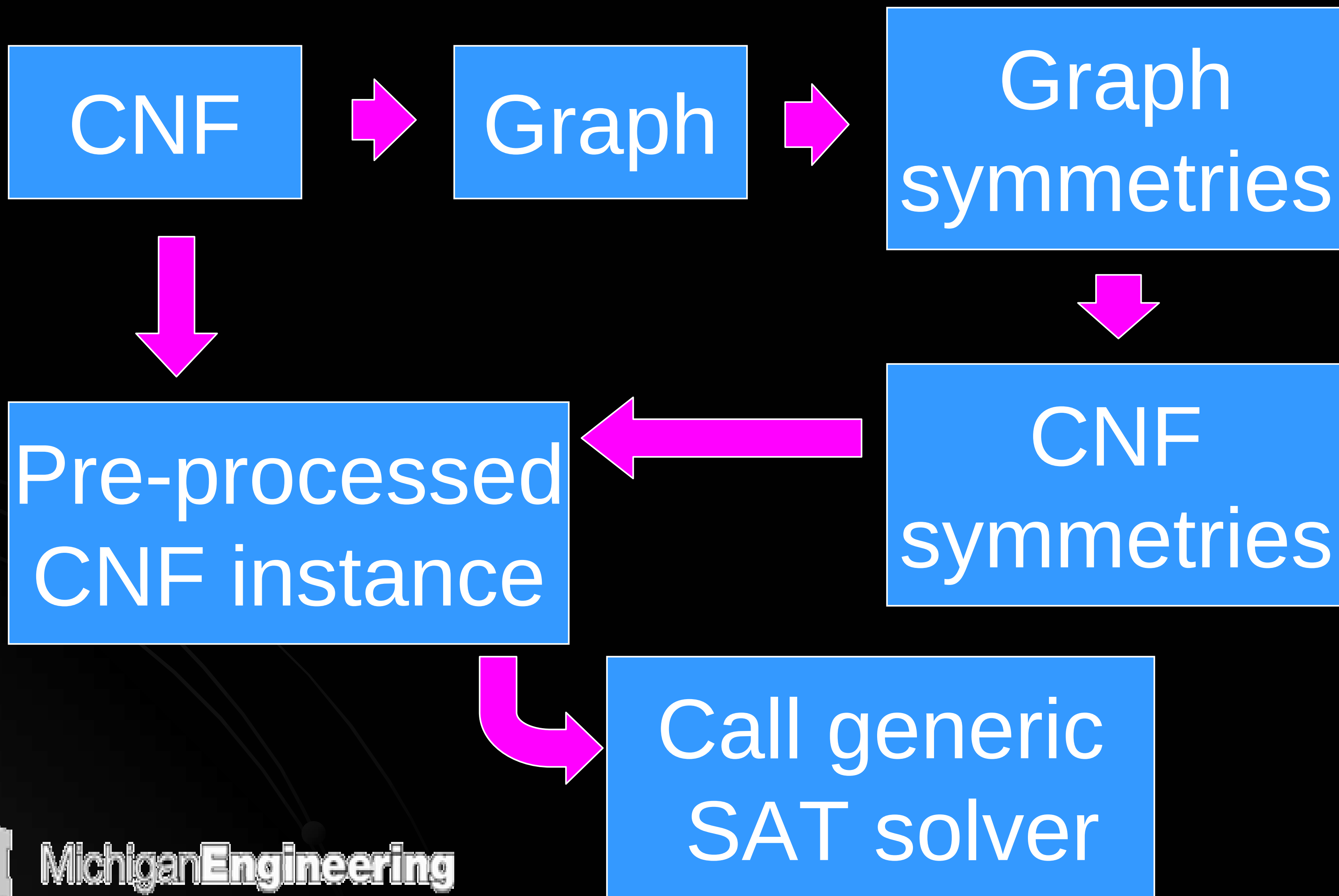
Motivation

- Exponential gap in proof lengths
  - The pigeon-hole principle (as a SAT instance)
  - Exponential lower bounds for *resolution* proofs
    - Beame, Karp and Pitassi, 2002: $\Omega(2^{n/20})$
  - Resolution + reasoning by symmetry $\rightarrow$ poly-sized proofs
    - Krishnamurthy, 1988
- Lower bounds on resolution proofs *apply* to the behavior of DP/DLL SAT solvers
- <u>One must also capture the complexity of symmetry extraction and the size of symmetry representation</u>

Goals of This Work

- **Push the envelope of constraint satisfaction**
  - Problem instances from apps have symms
- **Develop generic methods for CSPs**
  - For now, our focus is on Boolean Satisfiability
    - *This problem is fundamental to Math & CS*
  - Yet, we feel our work applies beyond SAT
    - 0-1 ILP and generic ILP are natural extensions
- **<u>High-performance, competitive</u> methods**
  - Must minimize overhead of dealing w symmetry

How We Do It



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Benchmarking

- Take a strong SAT solver, e.g., Chaff
- Run it on a CNF benchmark
- Run the proposed flow
- Compare runtimes
  - Plain SAT solver *versus*
  - Pre-processing + SAT solver
- No assumption about "symmetries being known" – this enables many applications

Why Pre-processing?

- Alternative: hack some SAT solver
  - More sophisticated strategies possible
    - *Ditto for publications* ☺
  - Can potentially capture “partial symmetries”
  - <u>Additional overhead</u>: is this really worth doing?
    - Li and Purdom, SAT 2002: *in some cases, yes*
- Do you really want to hack Chaff ?
  - Perhaps, but that does not prevent applying pre-processing first !

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Our Experience

- Our DAC 2002 paper shows that
  - **Pre-processing + Chaff** beats **plain Chaff** (*or is very close)* on realistic benchmarks
  - **Symmetry extraction time is significant**
    - In some cases **that** limits competitiveness
  - Sometimes there are very few symmetries

    → Symmety extraction isn't very useful and therefore must be fast
- There may be more room for improvement

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Mathematical Background(1)

- A symmetry (in a broad sense) of an object
  - Is a transformation that preserves its properties
- From Abstract Algebra
  - A **group** is a set with a binary operation on it
    - Must be associative
    - Must have a neutral element (unit)
    - Every element must have a (unique) inverse
- A **subgroup** is a subset closed under the op
  - Is a group by itself

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Mathematical Background(2)

- The Lagrange Theorem
  - The size of a finite group
    is divisible by the size of its subgroups
  - Corollary: **proper subgroups are ½ size or less**
- A **set of generators** of a group
  - Every group element is a product of generators
- For a group of size *N*, <u>an irredundant set of generators has no more than</u> $log_2 N$ <u>elements</u>

Relevant History

- Felix Klein studied symmetries of geometric shapes in the XIX century
- Group Theory was developed in XIX century as a formalism for capturing symmetries
  - Used much earlier, e.g., by Galois
  - Today is one of the major branches of Mathematics
- Ref: M. Hall Jr. "The Theory of Groups", 1959
- Symmetries are fundamental to modern physics
  - Quantum Mechanics and Relativity deal with groups of symmetries
- **It is seems natural to try using Group Theory in CS!**

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More Definitions

- Symmetries of a graph
  - Permutations of vertices that preserve edges
- **Symmetries of a SAT formula**
  - Permutations of variables that preserve clauses
  - Simultaneous negations of sets of vars that
    - “phase-shift” symmetries (auto-symmetries)
  - Compositions of the two types
- Can talk about *the symmetry group* of (i) a graph, (ii) a CNF formula

Computational Group Theory(CGT)

- Finite (and some infinite!) <u>groups</u> routinely <u>represented by generators</u>
- The **CGT** was in the works since 1900s, and flourished since 1960s
  - Reasonably efficient algorithms for perm groups (Sims, Knuth, Babai, others)
  - Excellent implementations available today (GAP)
- Graph Automorphism programs (NAUTY)

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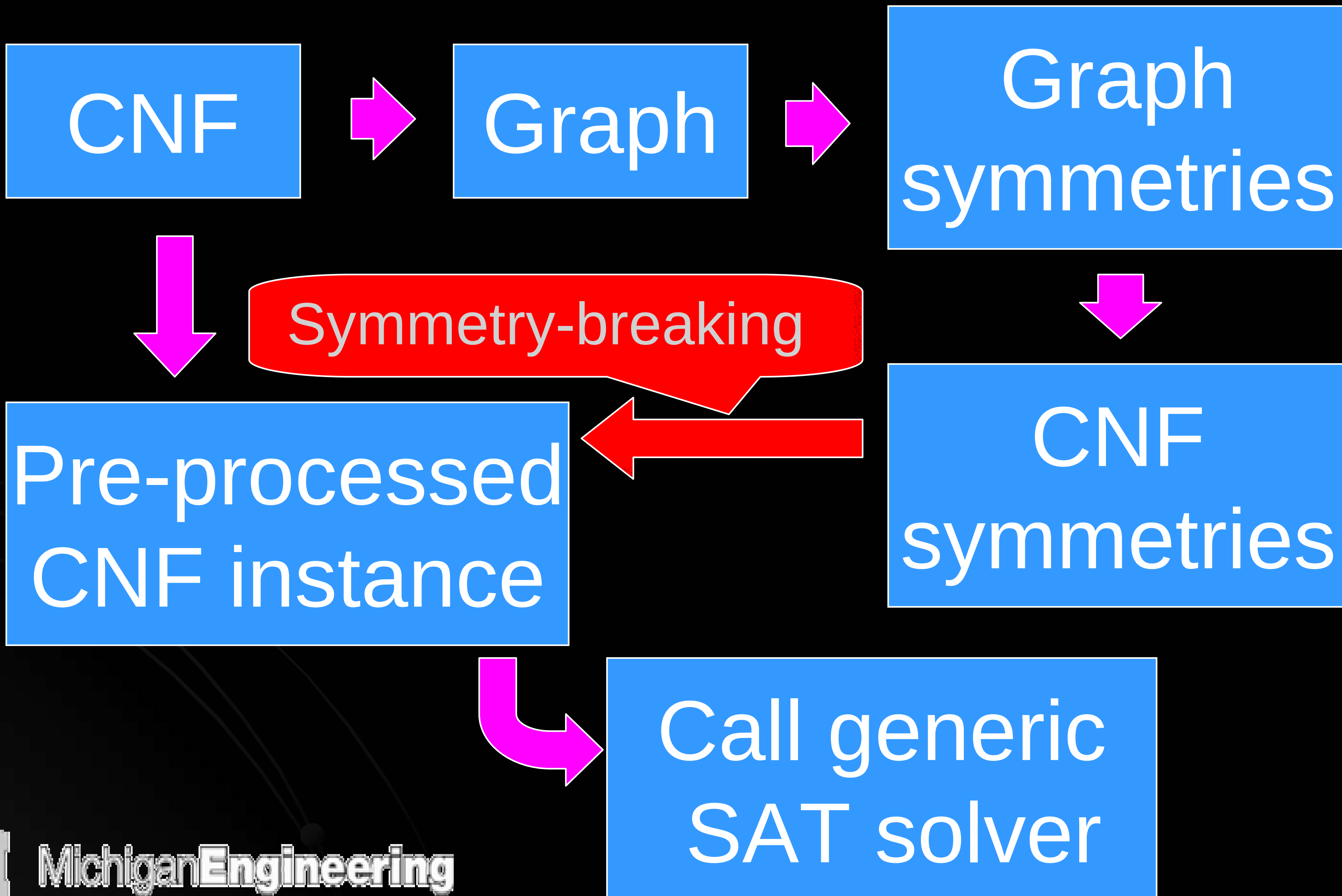
Using Symmetries in SAT



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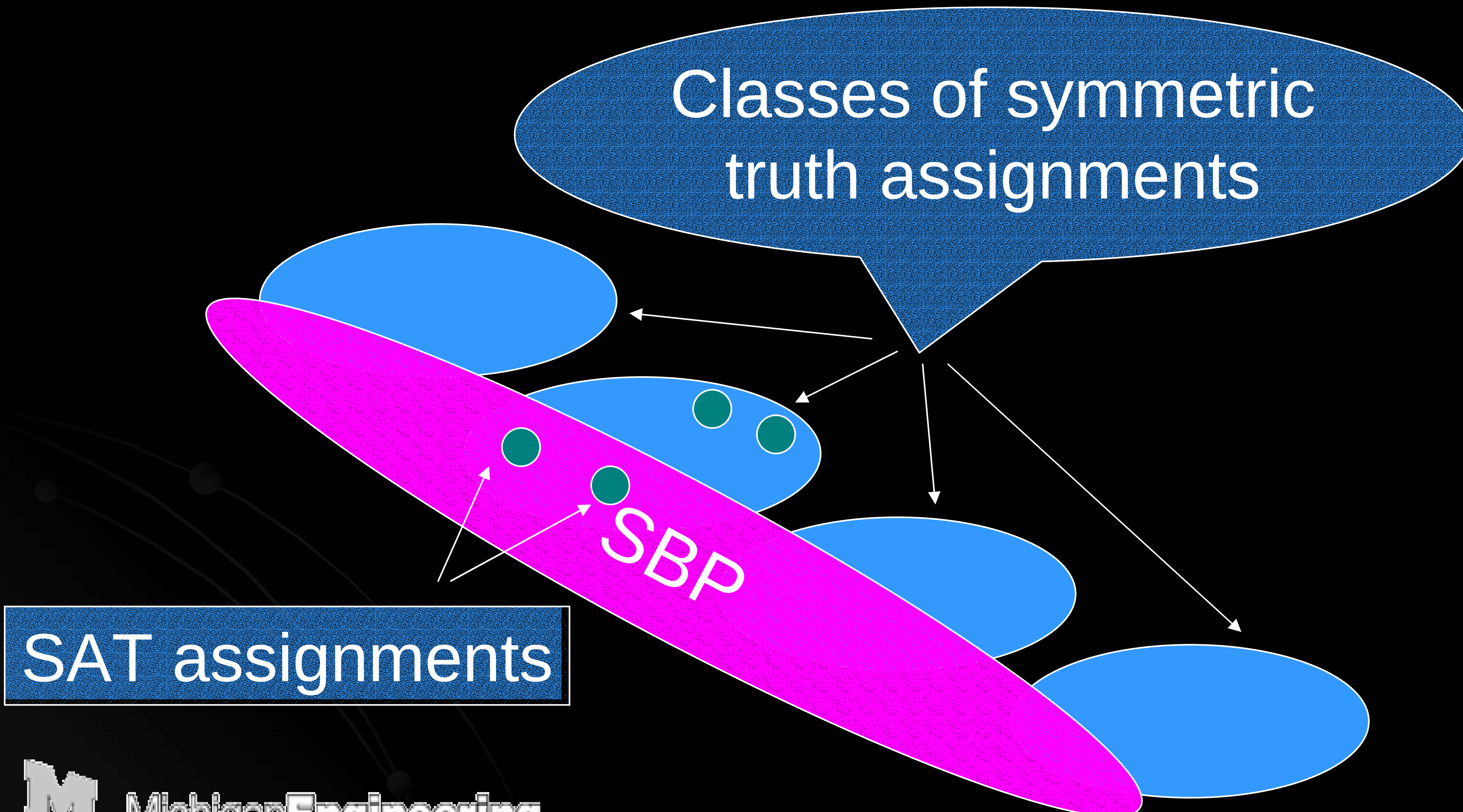
Symmetry-breaking Predicates

- A symmetry-breaking predicate (SBP) is what we add to a CNF formula ☺
  - to speed up DLL SAT solvers by pruning the search space
- **If a formula is satisfiable, a valid SBP must eval to TRUE on some SAT assignments**
- E.g., if *N* truth assignments are symmetric, an SBP may pick only one of them
  - We allow "<u>partial SBPs</u>" that pick *>1* solutions

Symmetry-Breaking Predicates



Previous Work (1)

- Crawford et al., "Symmetry-breaking Predicates For Search Problems", 1996
  - CNF symmetries via Graph Automorphism
  - <u>Full lex-leader SBPs from symmetries</u>
    - Rather impractical *per se*, but of fundamental value
  - The concept of a symmetry tree
    - Not used in our work
  - A discussion of examples, several ideas we use
  - No convincing empirical results

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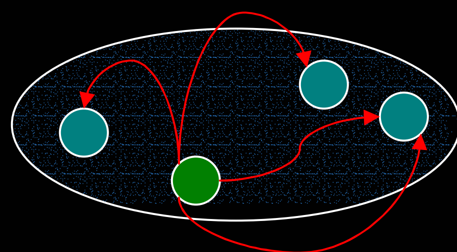
How to Select Lex-leaders

- Idea: select lexicographically smallest assignments from each equivalence class



- Crawford et al. construct an SBP for that:
  - map a given assignment by all symmetries
  - and require that every image be lex-greater

→ Conjunction over all symmetries ☹

Previous Work (2)

- Our recent papers *at SAT 2002, DAC 2002* and a 40-page Tech. Rep. at **http://satlive.org** **"Solving Difficult Instances of SAT in the Presence of Symmetry"**
  - Improved/corrected use of Graph Automorphism
  - SBPs in terms of cycles of permutations
  - Partial SBPs via generators of symmetry groups
  - Strong, detailed empirical results
  - Fast "opportunistic" symmetry extraction

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SBPs in cycle notation

- Suppose the variable ***z*** can be negated
  - Then we can add the SBP ***(z)***
- Suppose variables ***x*** and ***y*** can be swapped (with or w/o other variables being swapped)
  - Then we can add the SBP ($x \leq y$), i.e., ***(x'+y)***
- Similarly if ***x***, ***y*** and ***z*** can be permuted
  - We add the SBP ($x \leq y \leq z$), i.e., ***(x'+y) (y'+z)***
- Compared to Crawford et al, this is a form of logic minimization

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Contributions of This Work

- **Further improvements of SBPs** via logic minimization **(used in VLSI CAD)**
  - Economical SBPs $\rightarrow$ faster SAT-solving
  - Cases: single-cycles and multiple cycles
  - Another approach: direct improvement over Crawford
- **New, provable analyses** of partial symmetry-breaking by generators (**PSBG**)
- **A pitfall identified**: incompatible variable orderings
- **PSBG** for pigeon-holes is not complete
  - Yet, works extremely well in practice

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New Constructions of SBPs For Single Permutations

- Since permutations are represented in the cycle notation, we look at single cycles first
- **Then we chain multiple cycles**
- Important observation
  - The variable ordering and the chaining sequence must be compatible
  - This makes little difference for one permutation, but can spoil things for multiple permutations

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New Constructions of SBPs For Single Cycles (1)

- We show a counting formula for #classes of symmetric assignments under an *N*-cycle
  - First few numbers are: 3(**for 2-cycle**), 4, 6, 8
- The straightforward generalization from small cycles **(xy)** and **(xyz)** to **(xyzt)** does not yield a valid SBP!
- However, one can explicitly formulate this as a two-level logic minimization problem
  - Starting with a truth table, or
  - Starting with a CNF given by Crawford’s SBP

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New Constructions of SBPs For Single Cycles (2)

- We solve cycles of length<20 with ESPRESSO --- common software for two-level logic minimization
  - This gives a full lex-leader SBP for each *N*
  - We see some patterns, but no easy description
- We also propose a construction of partial lex-leader SBPs that works for any *N*
  - Can be used for very large cycles

SBPs for Multiple Cycles

- Lemma: **SBPs of cycles of co-prime lengths in the same permutation can be conjoined**
  - Proof: Each cycle is a power of the perm
- We give a more complex procedure for cycles whose lengths are not co-prime, e.g., (xyz)(abcdef) or (xyzt)(abcd)
- Prime factors of the cycle length matter!

Symmetry-breaking for Multiple Generators (1)

- Lemma 5.1 essentially says:
  Fully-breaking a given single symmetry
  is equivalent to fully breaking all of its powers
- Lemma 5.2 :
  If a truth assignment is a lex-leader of an
  equivalence class under a group G, then
  it is a lex-leader …under any subgroup of G

Symmetry-breaking for Multiple Generators (2)

- We now consider cyclic subgroups generated by each generator
- Lemma 5.3:
  A conjunction of lex-leader SBPs of sub-groups is a valid SBP,
  however it may not be a full SBP
- Corollary 5.6: Consider two perms with disjoint support. The conjunction of their lex-leader SBPs is a full lex-leader SBP

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On Variable Orderings

- When breaking symmetries by generators
  - **especially efficient SBPs can be built for each generator by changing the order of variables**
- However, **variable orders must be consistent for all generators**
- We build a consistency graph:
  - One vertex per generator
  - Connect generators whose *supports* intersect
- Find a maximal (or just large) independ. set

SBPs for the Pigeon-hole Principle

- To show that not all symmetries are broken by a partial lex-leader SBP
  - We give a satisfying truth assignment that is not a lex-leader (*for hole-2*)
- Recall: all holes and all pigeons are symm.
  - The symmetry group is $S_n \times S_{n+1}$
  - Consider a set of generators that is a Cartesian product of those in S_n and S_{n+1}
  - All generators map our assignment into > ones

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Conclusions and On-going work

- **Logic minimization leads to better SBPs**
- **Symmetry-breaking by generators is a sound and viable technique**
  - Yet does not provide full symmetry-breaking in some important cases
- On-going work
  - **Faster symmetry extraction**
    - Generic and specialized, complete and incomplete
  - **Further improvements of SBPs**
  - **Going beyond pre-processing**

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