Using Graph Properties for Global Constraints for Necessary Conditions and Filtering

Nicolas Beldiceanu[1], Mats Carlsson[2]
Jean-Xavier Rampon[3], Charlotte Truchet[3]
Sophie Demassey[1], Thierry Petit[1]

École des Mines de Nantes[1] (EMN, LINA)
Swedish Institute of Computer Science[2] (SICS)
Université de Nantes[3] (UN, LINA)

March 14, 2006

Outline

Nicolas Beldiceanu, Mats Carlsson, Jean-Xavier Rampon, Charlotte Truchet, Sophie Demassey, Thierry Petit

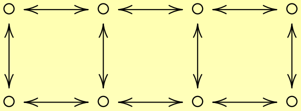
Context and Key Ideas

- Global Constraints as Graph Properties of Structured Networks of Elementary Constraints of the Same Type [BelCarRam05].
- Graph Properties are not independent. They are related by Graph Invariants.
- Graph Invariants are generic. Some 150 of them have been collected in a database.
- Given a constraint C specified in terms of Graph Properties, the relevant Graph Invariants form necessary conditions for C.
- Bounds on Graph Characteristics can be computed dynamically and be used for pruning.

Nicolas Beldiceanu, Mats Carlsson, Jean-Xavier Rampon, Charlotte Truchet, Sophie Demassey, Thierry Petit

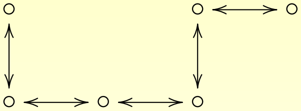
A Simple Global Constraint

Initial network



Arcs are associated with elementary constraints.

Final network

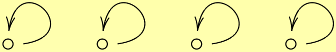


Ask properties of sub-graph of elementary constraints that still hold.

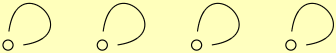
Nicolas Beldiceanu, Mats Carlsson, Jean-Xavier Rampon, Charlotte Truchet, Sophie Demassey, Thierry Petit

Graph Generators

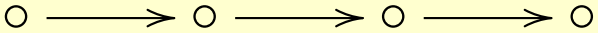
LOOP



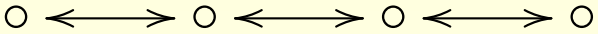
SELF



PATH

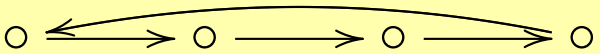


CHAIN

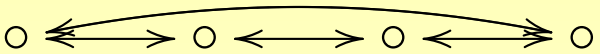


Graph Generators

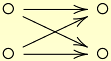
CIRCUIT



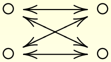
CYCLE



PRODUCT

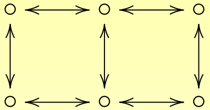


SYMMETRIC PRODUCT

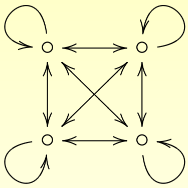


Graph Generators

GRID



CLIQUE



Nicolas Beldiceanu, Mats Carlsson, Jean-Xavier Rampon, Charlotte Truchet, Sophie Demassey, Thierry Petit

SWEDISH INSTITUTE OF COMPUTER SCIENCE SICS
ECOLE DES MINES DE NANTES
UNIVERSITÉ DE NANTES
LABORATOIRE D'INFORMATIQUE DE NANTES ATLANTIQUE lina

Graph Characteristics

NVERTEX	$\|V(G)\|$
NEDGE	$\|E(G)\|$
NSOURCE	number of vertices without predecessor
NSINK	number of vertices without successor
NCC	number of connected components of G
MIN_NCC	number of vertices of smallest c.c. of G
MAX_NCC	number of vertices of largest c.c. of G
NSCC	number of strongly connected components of G
MIN_NSCC	number of vertices of smallest s.c.c. of G
MAX_NSCC	number of vertices of largest s.c.c. of G

Nicolas Beldiceanu, Mats Carlsson, Jean-Xavier Rampon, Charlotte Truchet, Sophie Demassey, Thierry Petit

Graph Properties and Graph Invariants

- A graph property is a relation $C \circ V, \circ \in \{\leq, \geq, =, \neq\}$, where C is a graph characteristic and V is a domain variable.
- A graph invariant is a relation on graph characteristics that is valid for a graph class.
- Example:

$$\mathrm{MIN\_NSCC} \neq \mathrm{MAX\_NSCC}$$
$$\Rightarrow$$
$$\mathrm{NVERTEX} \geq \mathrm{MIN\_NSCC} + \mathrm{MAX\_NSCC}$$

Nicolas Beldiceanu, Mats Carlsson, Jean-Xavier Rampon, Charlotte Truchet, Sophie Demassey, Thierry Petit

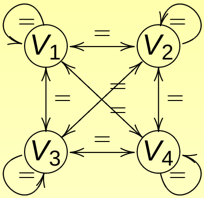
nvalue($NVAL$, $VARS$)

arguments	$NVAL$: dvar, $VARS$: collection(var − dvar)
restrictions	$0 \leq NVAL \leq \|VARS\|$
arc input	variables
arc generator	clique
arc constraint	$VARS$.var[1] = $VARS$.var[2]
graph properties	NSCC = $NVAL$
example	nvalue(3, {var − 3, var − 1, var − 7, var − 1})

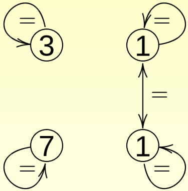
Nicolas Beldiceanu, Mats Carlsson, Jean-Xavier Rampon, Charlotte Truchet, Sophie Demassey, Thierry Petit

nvalue$(3, \{var-3, var-1, var-7, var-1\})$

Initial network: variables unbound



Final network: variables instantiated, $NSCC = 3$



Graph Invariants for nvalue

A lower bound on *NVAL* in nvalue(*NVAL*, *VARS*):

$$\mathrm{NSCC} \geq \left\lceil \frac{\mathrm{NVERTEX}^2}{\mathrm{NARC}} \right\rceil$$

Nicolas Beldiceanu, Mats Carlsson, Jean-Xavier Rampon, Charlotte Truchet, Sophie Demassey, Thierry Petit

Tighter Graph Invariants

- Typically, the graph for a global constraint has a specific structure. The arc generator and arc constraint determine the graph class.
- A general graph invariant:

$$\mathrm{NARC} \leq \mathrm{NVERTEX}^2$$

- A tighter graph invariant that holds for graph class PATH:

$$\mathrm{NARC} \leq \mathrm{NVERTEX} - 1$$

- Other invariants are specific e.g. for acyclic, bipartite, or symmetric graphs.

A Database of Graph Invariants

Queried by: a set of graph characteristics (GCs) and a graph class, determined by the constraint of interest.
Statistics:

#graphs	#GC	#invariants
1	1	13
1	2	50
1	3	34
1	4	12
1	5	2
2	2	10
2	3	10
2	4	6
2	5	16
2	6	4

Nicolas Beldiceanu, Mats Carlsson, Jean-Xavier Rampon, Charlotte Truchet, Sophie Demassey, Thierry Petit

SWEDISH INSTITUTE OF COMPUTER SCIENCE SICS
ECOLE DES MINES DE NANTES
UNIVERSITÉ DE NANTES
LABORATOIRE D'INFORMATIQUE DE NANTES ATLANTIQUE lina

Bounds on Graph Characteristics

- Results to date on general graphs are shown in the table.
- Tighter and cheaper bounds can be found for specific graph classes.

G.C.	Sharp	Complexity	Bound
$\underline{\mathrm{NARC}}$	yes	P	$\lvert E_T\rvert + \lvert X_{T,\neg T}\rvert - \mu(\overleftrightarrow{G}(X_{T,\neg T}, E_U))$
$\overline{\mathrm{NARC}}$	yes	P	$\lvert E_{TU}\rvert$
$\underline{\mathrm{NVERTEX}}$	yes	NP	$\lvert X_T\rvert + h(\overleftrightarrow{G}((X_{T,\neg T,\neg T}, X_{U,\neg T,T}), E_{U,T}))$
$\overline{\mathrm{NVERTEX}}$	yes	P	$\lvert X_{TU}\rvert$
$\underline{\mathrm{NCC}}$	yes	P	$\lvert cc_{[\lvert X_T\rvert \geq 1]}(\overrightarrow{G}(X_{TU}, E_{TU}))\rvert$
$\overline{\mathrm{NCC}}$	yes	P	$\lvert cc_{[\lvert E_T\rvert \geq 1]}(\overrightarrow{G}(X_T, E_T))\rvert + \mu_I(\overleftrightarrow{G}_{rem})$
$\underline{\mathrm{NSCC}}$	yes	NP	$\lvert scc_{[\lvert X_T\rvert \geq 1]}(\overrightarrow{G}(X_{TU}, E_{TU}))\rvert + h(G_{\underline{\mathrm{NSCC}}}((Y, Z), E))$
$\overline{\mathrm{NSCC}}$	yes	P	$\lvert scc(\overrightarrow{G}(X_{TU}, E_T))\rvert$
$\underline{\mathrm{NSINK}}$	yes	NP	$\lvert sink_{[\lvert X_T\rvert = 1]}(\overrightarrow{G}(X_{TU}, E_{TU}))\rvert + h(G'_r((Y, Z), E))$
$\overline{\mathrm{NSINK}}$	no	P	$\lvert sink(\overrightarrow{G}(X_T, E_T))\rvert + \lvert X_U\rvert - \lvert source_{[\lvert X_U\rvert = 1]}(\overrightarrow{G}(X_{TU}, E_{TU}))\rvert - \lvert XP\rvert$

Filtering: definitions

Given a constraint $C(v_1, \ldots, v_n,\ x_1, \ldots, x_m)$ with associated digraph $\mathcal{G} = (\mathcal{X}, \mathcal{E})$, binary arc constraint *ctr*, graph characteristics $\Xi_1, \ldots, \Xi_n$, and variables:

- A 0/1 variable z_j for each vertex $j \in \mathcal{X}$.
- A 0/1 variable z_{jk} for each arc $(j, k) \in \mathcal{E}$.

C is equivalent to the following system of constraints:

$$z_{jk} = 1 \ \Leftrightarrow ctr(x_j, x_k),\ (j, k) \in \mathcal{E} \qquad (1)$$

$$z_j = \bigvee_{\{k \mid (j,k) \in \mathcal{E} \vee (k,j) \in \mathcal{E}\}} z_{jk},\ j \in \mathcal{X} \qquad (2)$$

$$c_i = \Xi_i(\{z_j \mid j \in \mathcal{X}\}, \{z_{jk} \mid (j, k) \in \mathcal{E}\}),\ 1 \leq i \leq n \qquad (3)$$

$$c_i \circ_i v_i,\ 1 \leq i \leq n \qquad (4)$$

Graph-Based Filtering: a first attempt

- Given a constraint $C(v_1, \ldots, v_n,\ x_1, \ldots, x_m)$, filtering can be obtained by posting constraints (1,2,3,4).
- Constraints (3) need propagators:

PROCEDURE $\Xi(\{z_j \mid j \in \mathcal{X}\}, \{z_{jk} \mid (j,k) \in \mathcal{E}\}, c)$

1: Evaluate $\underline{c'}$ and $\overline{c'}$ wrt. $(\{z_j\}, \{z_{jk}\})$
2: $\min(c) \leftarrow \max(\underline{c'}, \min(c))$
3: $\max(c) \leftarrow \min(\overline{c'}, \max(c))$
4: **if** $\min(c) = \max(c) = \overline{c'}$ **then**
5: Fix some z_j, z_{jk} in order to avoid $c' < \overline{c'}$
6: **if** $\min(c) = \max(c) = \underline{c'}$ **then**
7: Fix some z_j, z_{jk} in order to avoid $c' > \underline{c'}$

Nicolas Beldiceanu, Mats Carlsson, Jean-Xavier Rampon, Charlotte Truchet, Sophie Demassey, Thierry Petit
SWEDISH INSTITUTE OF COMPUTER SCIENCE SICS
ECOLE DES MINES DE NANTES
UNIVERSITÉ DE NANTES
LABORATOIRE D'INFORMATIQUE DE NANTES ATLANTIQUE lina

group01 — a PATH+LOOP Constraint

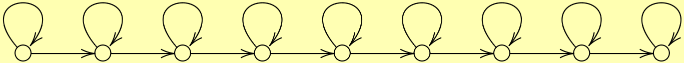
group01(*NGroup*, *MinSize*, *MaxSize*, *MinDist*, *MaxDist*, *NOne*, *VARS*) holds if:

- *VARS* is a sequence of 0/1-variables
- an *i-group* is a maximal sequence of values *i*
- *VARS* contains *NGroup* 1-groups
- *MinSize* (*MaxSize*) is the length of the smallest (largest) 1-group
- *MinDist* (*MaxDist*) is the length of the smallest (largest) 0-group
- *NOne* is the total number of 1s

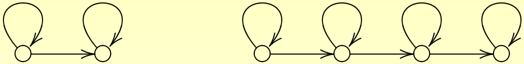
group01 — Graph Properties

$\mathrm{group01}(2, 2, 4, 1, 2, 6, \{0, 0, 1, 1, 0, 1, 1, 1, 1\})$

Initial network: variables unbound



Final network: ones



Final network: zeros

Nicolas Beldiceanu, Mats Carlsson, Jean-Xavier Rampon, Charlotte Truchet, Sophie Demassey, Thierry Petit

group01 — Filtering

group01($NGroup, MinSize, MaxSize, MinDist, MaxDist,$ $NOne, VARS$) with m 0/1-variables is equivalent to:

$$z_j = (VARS_j \wedge VARS_{j+1}),\ 1 \leq j < m \quad (5)$$

$$NGroup = \mathrm{NCC}(VARS, \{z_j\}) \quad (6)$$

$$MinSize = \mathrm{MIN\_NCC}(VARS, \{z_j\}) \quad (7)$$

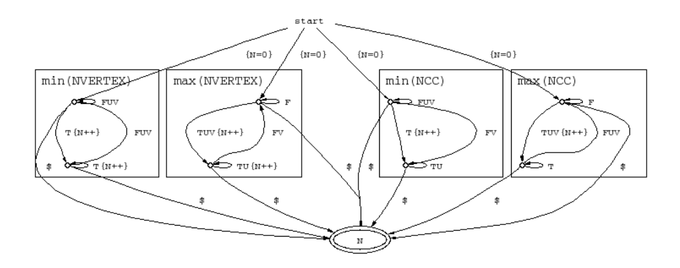
$$MaxSize = \mathrm{MAX\_NCC}(VARS, \{z_j\}) \quad (8)$$

$$MinDist = \mathrm{MIN\_NCC}_C(VARS, \{z_j\}) \quad (9)$$

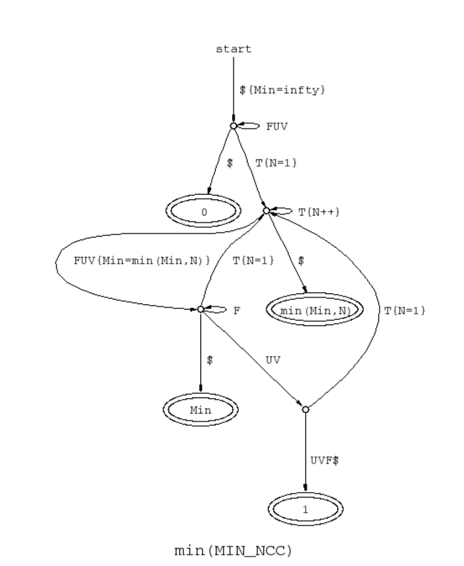
$$MaxDist = \mathrm{MAX\_NCC}_C(VARS, \{z_j\}) \quad (10)$$

$$NOne = \mathrm{NVERTEX}(VARS, \{z_j\}) \quad (11)$$

Bounds on graph characteristics for PATH+LOOP

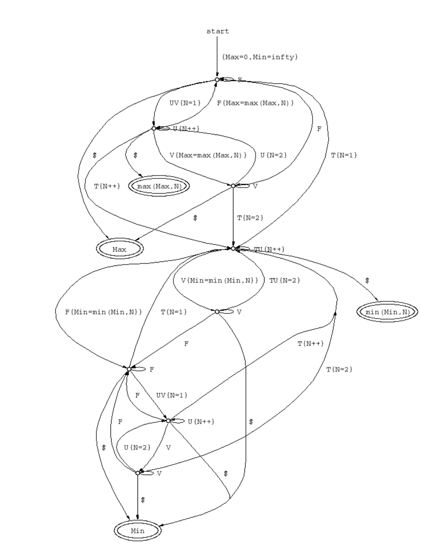


Bounds on graph characteristics for PATH+LOOP



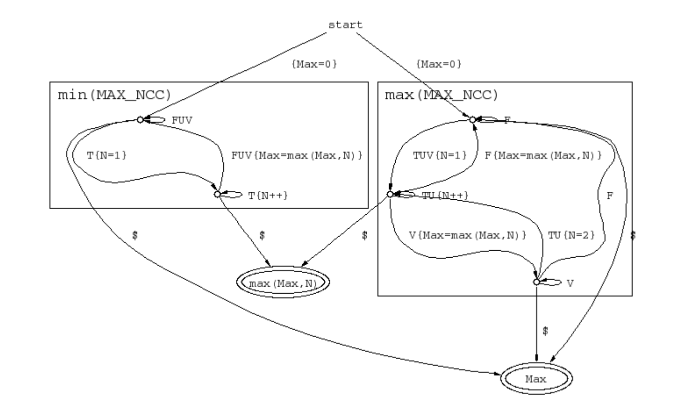
Nicolas Beldiceanu, Mats Carlsson, Jean-Xavier Rampon, Charlotte Truchet, Sophie Demassey, Thierry Petit

Bounds on graph characteristics for PATH+LOOP



max(MIN_NCC)

Bounds on graph characteristics for PATH+LOOP



Some Graph-Based Filtering for PATH+LOOP

Let *U* be a maximal sequence of nonground vertices joined by nonzero arcs. If $\mathrm{dom}(\mathrm{NCC}) = \{\underline{\mathrm{NCC}}\}$ then:

1. Any *U* neighboring *two* 1-vertices is assigned to a sequence of 1s.
2. Any *U* neighboring *no* 1-vertex is assigned to a sequence of 0s.

Some Graph-Based Filtering for PATH+LOOP

Let U be a maximal sequence of nonground vertices.
If $\mathrm{dom}(\mathrm{NCC}) = \{\overline{\mathrm{NCC}}\}$ then:

1. Within any U, z_j are assigned to 0.
2. Any U with odd $|U|$ neighboring *two* 1-vertices is assigned to an alternating sequence $0, 1, \ldots$.
3. Any U with even $|U|$ preceded by *one* 1-vertex is assigned to an alternating sequence $0, 1, \ldots$.
4. Any U with even $|U|$ succeeded by *one* 1-vertex is assigned to an alternating sequence $1, 0, \ldots$.
5. Any U with odd $|U|$ neighboring *no* 1-vertex is assigned to an alternating sequence $1, 0, \ldots$.

Conclusion

- The view of Global Constraints as Graph Properties of Structured Networks of Elementary Constraints of the Same Type is more than just a catalog.
- Generic invariants among the non-independent graph properties for a constraint C can be looked up automatically and give rise to necessary conditions.
- Bounds on Graph Characteristics can be computed dynamically and be used for pruning, allowing us to get a filtering scheme from a declarative description of a global constraint.

Nicolas Beldiceanu, Mats Carlsson, Jean-Xavier Rampon, Charlotte Truchet, Sophie Demassey, Thierry Petit

References

Nicolas Beldiceanu.
Global Constraints as Graph Properties on a Structured Network of Elementary Constraints of the Same Type.
Proc. CP'2000, LNCS 1894, 2004.

Nicolas Beldiceanu, Mats Carlsson, Jean-Xavier Rampon.
Global Constraint Catalog.
SICS Technical Report T2005-08, 2005.

References

- Nicolas Beldiceanu, Mats Carlsson, Thierry Petit.
  Deriving filtering algorithms from constraint checkers.
  Proc. CP'2004, LNCS 3258, 2004.
- Nicolas Beldiceanu, Mats Carlsson, Jean-Xavier Rampon, Charlotte Truchet.
  Graph Invariants as Necessary Conditions for Global Constraints.
  Proc. CP'2005, LNCS 3709, 2005.
- Nicolas Beldiceanu, Thierry Petit, G. Rochard.
  Bounds of Graph Characteristics.
  Proc. CP'2005, LNCS 3709, 2005.

Nicolas Beldiceanu, Mats Carlsson, Jean-Xavier Rampon, Charlotte Truchet, Sophie Demassey, Thierry Petit