Practically Feasible Proof Logging for Pseudo-Boolean Optimization

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SAT and Combinatorial Solving

- Impressive progress in SAT solving over last couple of decades [BHvMW21]
- Also big successes in more expressive paradigms:
  - Constraint programming
  - Satisfiability modulo theories (SMT)
  - Mixed integer linear programming (MILP)

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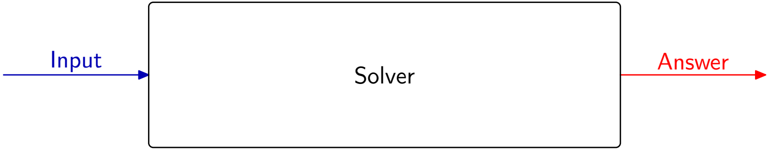
- However, solvers are sometimes wrong (even best commercial ones) [CGS17, AGJ+18, GSD19, GCS23, BBN+23, Tin24]
- Most successful solution: certifying algorithms with proof logging

Proof Logging with Certifying Solvers: Workflow



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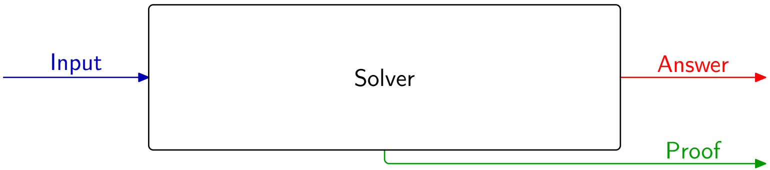

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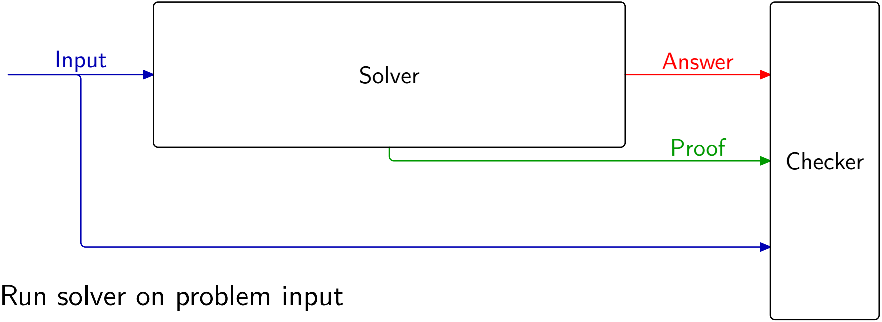

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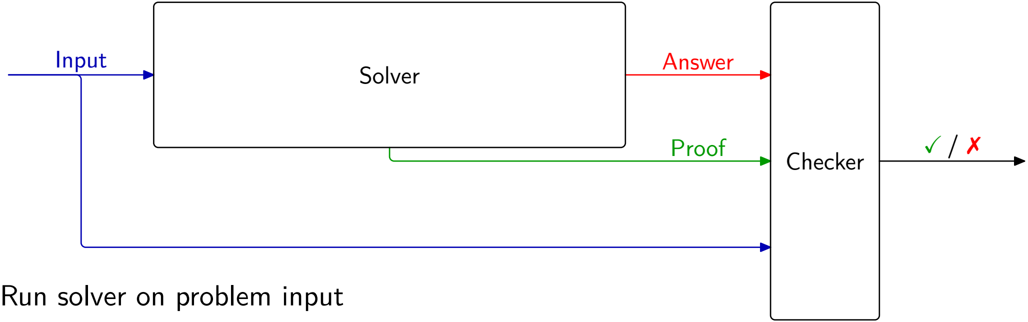

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3. Feed input + answer + proof to proof checker
4. Verify that proof checker says answer is correct

Proof Logging for SAT Solving and Beyond

- Proof logging for SAT: big success story [HHW13, WHH14, CHH+17, BCH21]
- Great performance:
  - Small constant overhead for proof generation ($\lessapprox 10\%$ of solving time)
  - Efficient proof checking ($\lessapprox 10\times$ solving time)

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  - Efficient proof checking ($\lesssim 10\times$ solving time)
- More expressive paradigms: VeriPB proof logging
  - MaxSAT solving [VDB22, BBN+23, IOT+24, BBN+24]
  - Subgraph solving [GMN20, GMM+20, GMM+24]
  - Constraint programming [EGMN20, GMN22, MM23, MMN24, MM25]
  - Automated planning [DHN+25]
  - and more ...

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  - Automated planning [DHN+25]
  - and more ...
- But much worse performance:
  - Proof logging overhead: sometimes $\times 10$ or worse
  - Proof checking overhead: sometimes $\times 1000$ or worse

Our Contribution

- Efficient VeriPB proof logging and checking for pseudo-Boolean optimization
- Covers all techniques in state-of-the-art solvers RoundingSat and Sat4j
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- Efficient VeriPB proof logging and checking for pseudo-Boolean optimization
- Covers all techniques in state-of-the-art solvers RoundingSat and Sat4j
- Including formally verified proof checking backend
- Performance close to expectations for SAT solving:
  - Proof logging overhead usually $\leqslant 10\%$ (worst-case 50%)
  - Checking overhead usually $\leqslant \times 6$ (worst-case $\times 20$)
- First time practically feasible logging for combinatorial optimization beyond SAT

Pseudo-Boolean Optimization

- Operates on 0-1 integer linear inequalities or pseudo-Boolean constraints:

$$\sum_i a_i \ell_i \geq A$$

  - $a_i, A \in \mathbb{Z}$
  - literals ℓ_i: x_i or $\overline{x}_i$ (where $x_i + \overline{x}_i = 1$)
  - variables x_i take values $0 = \mathit{false}$ or $1 = \mathit{true}$

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- Examples of pseudo-Boolean constraints:
  - Clauses: $x_1 \vee x_2 \vee \overline{x_3} \quad \Longleftrightarrow \quad x_1 + x_2 + \overline{x_3} \geq 1$
  - Cardinality constraints: $x_1 + x_2 + x_3 \geq 2$
  - General constraints: $3x_1 + 2x_2 + x_3 + x_4 \geq 3$

Approaches for Pseudo-Boolean Optimization

- Two main approaches:
  - Translate to CNF and run conflict-driven clause learning (CDCL)
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- Two main approaches:
  - Translate to CNF and run conflict-driven clause learning (CDCL)
  - Generalize conflict-driven search to pseudo-Boolean inequalities (our focus)
- New challenges and techniques compared to SAT:
  - Efficient propagation [Dev20, NORZ24]
  - Linear programming (LP) integration [DGN21]
  - Optimization techniques, e.g. solution-improving search, core-guided search [DGD+21]

Proof Logging for Pseudo-Boolean Optimization

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- Other techniques pose further challenges:
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  - Linear programming (LP) integration (Farkas certificates, cut generation, ...)

Proof Logging for Pseudo-Boolean Optimization

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  - In PB, explicit reasoning steps are needed
- Other techniques pose further challenges:
  - Objective rewriting in core-guided search
  - Linear programming (LP) integration (Farkas certificates, cut generation, ...)
- Challenges for efficient proof logging:
  - Logging unit constraints (saying that a variable must take some fixed value)
  - Logging constraint simplifications (e.g. simplifying away fixed values)
  - Logging and checking solutions
  - Formally verified proof checking

Pseudo-Boolean Proof Logging Basics

Pseudo-Boolean proof logging based on cutting planes proof system [CCT87]

Input axioms From the input

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Literal axioms $\overline{\ell_i \geq 0}$

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Input axioms — From the input

Literal axioms

$$\frac{}{\ell_i \geq 0}$$

Addition

$$\frac{\sum_i a_i \ell_i \geq A \qquad \sum_i b_i \ell_i \geq B}{\sum_i (a_i + b_i)\ell_i \geq A + B}$$

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Division for any $c \in \mathbb{N}^+$

$$\frac{\sum_i a_i \ell_i \geq A}{\sum_i \left\lceil \frac{a_i}{c} \right\rceil \ell_i \geq \left\lceil \frac{A}{c} \right\rceil}$$

Cutting Planes Toy Example

$$w + 2x + y \geq 2$$

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Multiply by 2 $$\frac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4}$$

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$$\text{Multiply by } 2 \; \dfrac{\dfrac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4} \qquad w + 2x + 4y + 2z \geq 5}{3w + 6x + 6y + 2z \geq 9} \; \text{Add}$$

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$$\frac{\overline{z} \geq 0}{2\overline{z} \geq 0}$$
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Divide by 3: $\dfrac{3w + 6x + 6y \geq 7}{w + 2x + 2y \geq 3}$

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By naming constraints by integers and literal axioms by the literal involved as

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such a calculation is written in the proof log in reverse Polish notation as

```
pol @C1 2 * @C2 + ~z 2 * + 3 d
```

Advanced Pseudo-Boolean Proof Logging

We need a rule for deriving non-implied constraints (e.g. introducing new variables)

Redundance-based strengthening ([BT19, GN21], inspired by [JHB12], simplified)

F and $F \cup \{C\}$ are equisatisfiable if there is a substitution ω (mapping variables to truth values or literals), called a witness, for which

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- Proof sketch: If α satisfies F but falsifies C, then $\alpha \circ \omega$ satisfies $F \cup \{C\}$
- In a proof, the implication needs to be efficiently verifiable — every $D \in (F \cup \{C\})\restriction_\omega$ should follow from $F \cup \{\neg C\}$ either
  1. "obviously" or
  2. by explicitly presented derivation

Redundance Rule: Example from Logging Core-Guided Search

Suppose we know $D \doteq x_1 + x_2 + x_3 \geq 2$.

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Suppose we know $D \doteq x_1 + x_2 + x_3 \geq 2$.
Want to introduce variable y_3 such that

$$x_1 + x_2 + x_3 = 2 + y_3, \quad \text{i.e.} \quad \begin{cases} C_1 & \doteq \; x_1 + x_2 + x_3 \leq 2 + y_3 \\ C_2 & \doteq \; x_1 + x_2 + x_3 \geq 2 + y_3 \end{cases}$$

using condition $F \cup \{\neg C\} \models (F \cup \{C\})\restriction_\omega$.

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  $\neg C_2 \doteq x_1 + x_2 + x_3 \leq 1 + y_3$ implies $C_1\restriction_\omega \doteq x_1 + x_2 + x_3 \leq 2$

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VeriPB:

```
red +1 x1 +1 x2 +1 x3 -1 y3 <= 2; y3 -> 1
red +1 x1 +1 x2 +1 x3 -1 y3 >= 2; y3 -> 0
```

Farkas Certificates

The constraints

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are unsatisfiable even over the reals.

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Farkas certificate: positive linear combination of constraints (and literal axioms, e.g. $\overline{x}_4 \geq 0 \doteq -x_4 \geq -1$) proving this:

$$C_1 + C_2 + 2C_3 + (\overline{x_4} \geq 0) + (x_2 \geq 0) \doteq 0 \geq 2$$

is a contradiction.

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VeriPB: `pol @C1 @C2 + @C3 2 * + ~x4 + x2 +`

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  - Add constraint (*cut*) implied by input formula
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  - Rational optimum $x_1 = x_2 = x_3 = \frac{1}{2}$

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$$
\begin{aligned}
C_1 &\doteq x_1 + x_2 \geq 1 \\
C_2 &\doteq x_1 + x_3 \geq 1 \\
C_3 &\doteq x_2 + x_3 \geq 1
\end{aligned}
$$

  - Rational optimum $x_1 = x_2 = x_3 = \frac{1}{2}$
  - Adding C_1, C_2 and C_3 yields $2x_1 + 2x_2 + 2x_3 \geq 3$

Cut Generation: Basics

- Cut generation:
  - Technique borrowed from MIP
  - Add constraint (*cut*) implied by input formula
  - Cuts away rational solution found by LP solver
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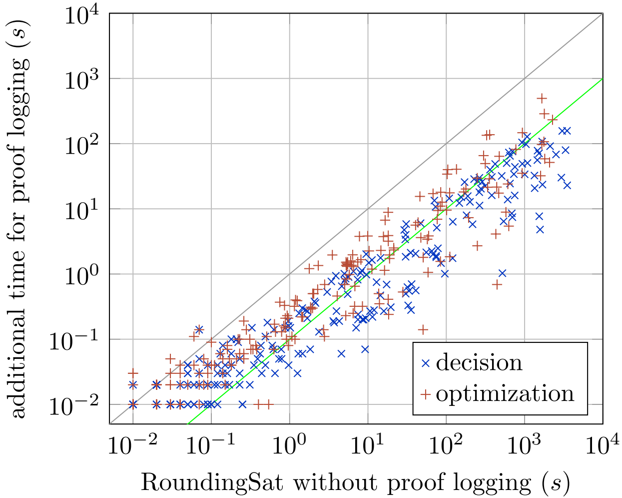
  - Cutting planes division by 2 yields $x_1 + x_2 + x_3 \geq 2$
  - VeriPB: `pol @C1 @C2 + @C3 + 2 d`

Advanced Cut Generation

- Cut generation with mixed integer rounding (MIR) rule [MW01, DGN21] more challenging
- Reasoning uses integer slack variables (not supported by VeriPB)
- Proof logging instead uses proof by contradiction

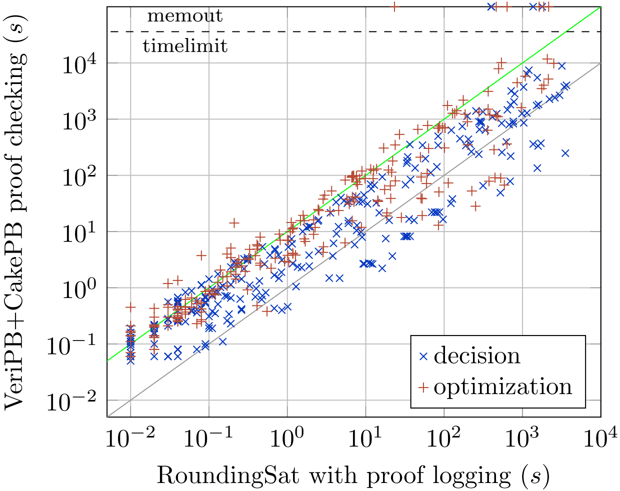
Empirical Results: Proof Logging Overhead

- Usually $\leqslant 10\%$
- Decision instances: worst-case 20%
- Optimization instances: worst-case 50%
- Overheads gets smaller for larger solving times



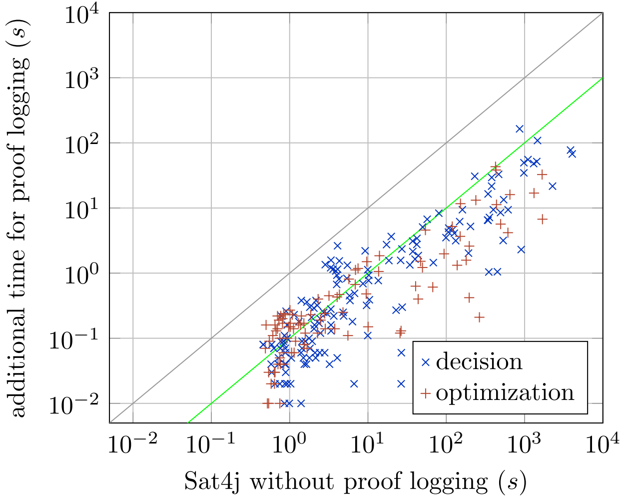
Empirical Results: Proof Checking Overhead

- Usually $\leqslant \times 6$
- Decision instances: worst-case $\times 10$
- Optimization instances: worst-case $\times 20$



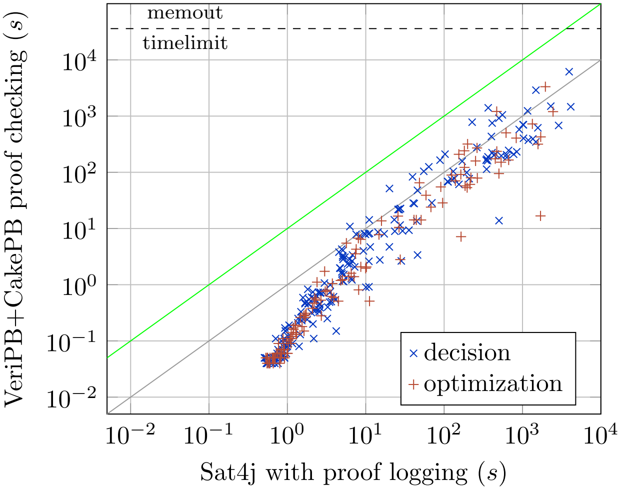
Empirical Results: Proof Logging Overhead Sat4j

- Usually $\leqslant 10\%$
- Worst-case 60%



Empirical Results: Proof Checking Overhead Sat4j

- Usually $\leqslant \times 2$
- Worst-case $\times 4$



Future Work

- Even faster proof logging and checking for pseudo-Boolean optimization
  - Branch-and-bound search (checking solutions currently a bottleneck)
  - Native efficient support for simplifications of constraints
  - Low-level optimizations in VeriPB and formally verified backend CakePB

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- Even faster proof logging and checking for pseudo-Boolean optimization
  - Branch-and-bound search (checking solutions currently a bottleneck)
  - Native efficient support for simplifications of constraints
  - Low-level optimizations in VeriPB and formally verified backend CakePB
- Faster proof logging and checking for further paradigms:
  - MaxSAT solving
  - Subgraph solving
  - Constraint programming
  - ...

Conclusion

- Efficient proof logging for pseudo-Boolean optimization using VeriPB
- First example of practically feasible certified solving beyond SAT
- Future directions:
  - Further improvements for pseudo-Boolean optimization
  - Efficient certified solving in other paradigms
- Is this the start of a new era: practically feasible proof logging beyond SAT?

Conclusion

- Efficient proof logging for pseudo-Boolean optimization using VeriPB
- First example of practically feasible certified solving beyond SAT
- Future directions:
  - Further improvements for pseudo-Boolean optimization
  - Efficient certified solving in other paradigms
- Is this the start of a new era: practically feasible proof logging beyond SAT?
- Thank you! Any questions?

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