Constraint solving on modular integers

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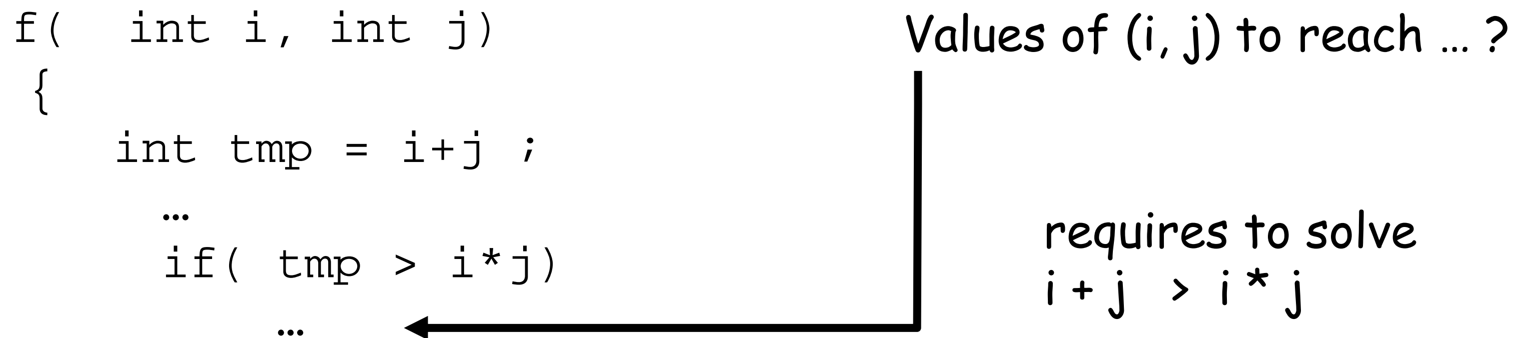
Software Verification with CP

- Automatic verification of programs (e.g., a C function or a Java method) requires the generation of test input that reach given locations



- *Constraint-Based Testing* tools include techniques that address this problem with:

  - CP over Finite Domains techniques
  - Abstract domains computations (Intervals, Polyhedra, Congruences, …)

Wrap-around integer computations

- Most architectures implement wrap-around arithmetic (modular integers) :

```
char (-128..127, 1 byte),                      unsigned char  (0..255, 1 byte),
short(-32768..32767, 2 byte),                  unsigned short (0..65535, 2 byte),
long (-2147483648..2147483647, 4 byte), unsigned long  (0.. 4294967295,4 byte),
…
```

- Problem in the previously mentionned tools:

Expressions are interpreted using ideal integer arithmetic rather than wrap-around integer arithmetic

- Example:

the C expression	`short a,b,c; c=a+b`	
should be interpreted as	c=a+b mod(2^{16})	in -32768..32767
rather than just	`c=a+b`	in inf .. sup

Programs that suppose wrap-around integer computations

- Good programming practices suggest taking care of integer overflows:

```
unsigned long len = 2^31;

int f( unsigned long buf ) {

        if (buf + len < buf)
                                  …
```

Value of buf to reach … ?

- Typical analysis tools would incorrectly declare … as being unreacheable !

NB: Simplifying `buf + len < buf` in `len < 0`
is forbidden in wrap-around integer arithmetic!

Bound-consistency for integer computations

Let `a,b` be unsigned over 4 bits

```
a in 0..15, b in 0..15
b = 2 * a;
```

```
// Ideal Arithmetic
// a in 0..7    b in 0..14
```

```
// Wrap-around arithmetic
// a in 0..15    b in 0..14
```

Bound-consistency for integer computations

Let `a,b` be unsigned over 4 bits

```
a in 8..9, b in 0..15
b = 2 * a;
```

```
// Ideal Arithmetic
// fail
```

```
// Wrap-around arithmetic
// a in 8..9   b in 0..2
```

Can we implement wrap-around interval ideal arithmetic with modulo ?

- Yes, but results wouldn't be optimal

  `A = 8, B in 2..4, C #= A*B mod(16)` (in SICStus clpfd)

  gives `C in 0..15` although `C=9,C=10, ..C=15` have no support !

- `8 * 2..4 =` $\{8*2=0_{16},\ 8*3=8_{16},\ 8*4=0_{16}\}$

  $\subset$ `0..8`

  smallest interval that contains all the double products!

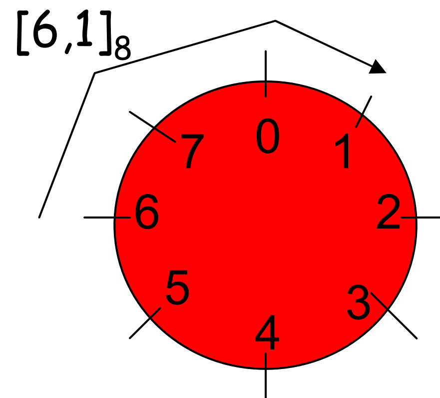
Our approach: to build an Interval Constraint Solver using *Clockwise Intervals*

<u>Def 1:</u> ***Clockwise Interval*** (CI)

Let $b=2^{\omega}$, x and y be two integers modulo b,

a CI $[x,y]_b$ denotes the set {x, x+1 mod b, .., y-1 mod b, y}

<u>Ex:</u> $[6,1]_8$ denotes the unordered set of integers modulo 8: {6,7,0,1}



By convention: $[0, b-1]_b$ is the canonical representation of Z_b

Cardinality

Def 2: ***Cardinality***

Let $[x,y]_b$ be a CI, then $card([x,y]_b)$ is an integer such that:

$$card([x,y]_b) = b \quad \text{if } [x,y]_b = [0, b-1]_b$$
$$= (y - x + 1) \bmod b \quad \text{otherwise}$$

Prop 1: A CI $[x,y]_b$ contains exactly $card([x,y]_b)$ elements

Hull

- The hull of a set of modular integers S is the smallest CI w.r.t. cardinality, that contains all the elements of S.

Def 3: ***(Hull)*** Let $S = \{x_1, .., x_p\}$ be a subset of Z_b, the hull of S is a CI, noted □S, □S = $Inf_{card}(\{[x_i,x_j]_b\} \mid \{x_1,..,x_p\} \subseteq [x_i,x_j]_b)$

Prop 2: Let $S = \{x_1, .., x_p\}$ be an **ordered** subset of Z_b, and let x_{-1} denotes x_{p-1}, then

□S = $[x_i,x_{i-1}]$ where i such that $card([x_i,x_{i-1}])$ is minimized

Corollary: □S can be computed in linear time w.r.t. the size of S

Clockwise interval arithmetic

$[i,j]_b$ @ $[k,l]_b$ = □{(i @ k) mod b, (i @ k+1) mod b, … (j @ l) mod b}

for any @ in {⊕, Θ, ⊗, …}

(Addition)

$[i,j]_b \oplus [k,l]_b = [0, b-1]_b$ if $card([i,j]_b)=b$ or $card([k,l]_b)=b$ or $card([i,j]_b)+card([k,l]_b \geq b$

$= [(i+k) \bmod b, (j+l) \bmod b]_b$ otherwise

(Substraction)

$[i,j]_b \Theta [k,l]_b = [0, b-1]_b$ if $card([i,j]_b)=b$ or $card([k,l]_b)=b$ or $card([i,j]_b)+card([k,l]_b \geq b$

$= [(i-l) \bmod b, (j-k) \bmod b]_b$ otherwise

Where the things become more complicated !

- Multiplication by a constant : $k \otimes [i,j]_b$
- Unlike in classical Interval Arithmetic,
  results cannot be computed using only the bounds

  $5 \otimes [2,7]_8 = \square\{10 \bmod 8, \dots, 35 \bmod 8\} = [1, 7]_8$

- but, 1) in Z_{2^w}, divisors of 0 are well-known

  2) Thanks to prop2, $\square\{x_1, .., x_p\}$ can be computed efficiently
  when $\{x_1, .., x_p\}$ is ordered

- Prop3: Let $k \neq 2^n$, q1 = k*i div b, q2 = k*j div b, then

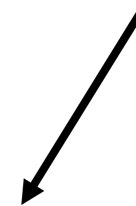
  $Max(k \otimes [i,j]_b) = b-d$ where $d = Min_{q1 < q \leq q2} (q*b \bmod k)$

  and

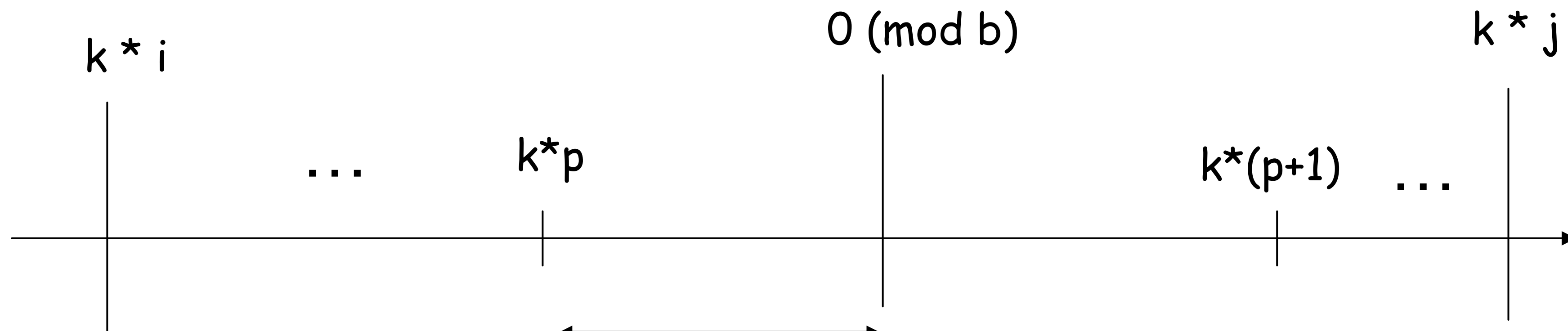
  $Min(k \otimes [i,j]_b) = d'$ where $d' = Min_{q1 < q \leq q2} (- q*b \bmod k)$

For $k * [i, j]_b$

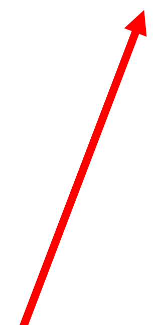
computing the upper bound can be done modulo k instead of modulo b !



$q*b = q*2^w = 0 \bmod b$



$d = \mathrm{Min}_{q1<q\leq q2} (q*b - k*p)$



$\mathrm{Max}(k * [i,j]_b)$

Then, $d = \mathrm{Min}_{q1<q\leq q2} (q*b \bmod k)$
and
$\mathrm{Max}(k \otimes [i,j]_b) = b-d$

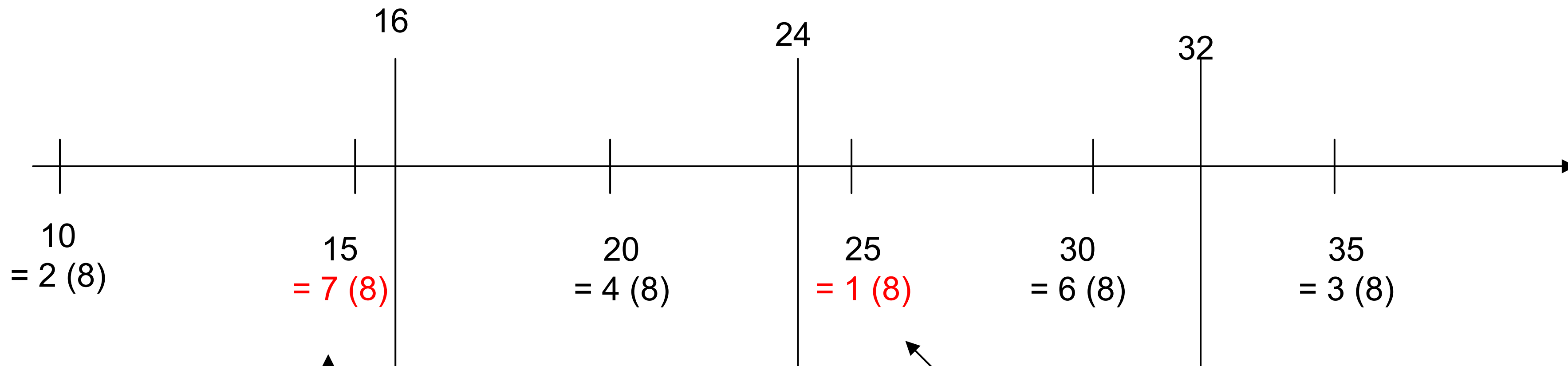
Prop3: Let $k \neq 2^n$, $q1$ = k*i div b, $q2$ = k*j div b, then

$Max(k \otimes [i,j]_b) = b-d$ where $d = Min_{q1< q \leq q2} (q*b \bmod k)$

and $Min(k \otimes [i,j]_b) = d'$ where $d' = Min_{q1< q \leq q2} (- q*b \bmod k)$

k=5, i=2, j=7, b=8

$5 * [2,7]_8 = [1,7]_8$



$d = Min_{q1<q\leq q2} (q*b \bmod k)$, $d'=Min_{q1<q\leq q2} (- q*b \bmod k)$,

q = 2, 16 mod 5 = 1, -16 mod 5 = 4

q = 3, 24 mod 5 = 4, -24 mod 5 = 1

q = 4, 32 mod 5 = 2, -32 mod 5 = 3

Relations over Clockwise Intervals

- Inclusion, union and intersection of CIs are defined with their set-theoretic counterparts

  $[i,j]_b \subseteq [k,l]_b \Leftrightarrow \{i,i+1,\ldots,j\} \subseteq \{k, k+1, \ldots, l\}$

- However, union and ***more surprisingy intersection*** are not closed over CIs, e.g.,

  $[5, 2]_8 \cap [1, 6]_8 = \{1, 2, 5, 6\}$

  Hence, we define the meet and join operations using the hull operator

  $[5, 2]_8$ meet $[1, 6]_8 = \square\{1, 2, 5, 6\} = [1,6]_8$

- X = Y leads to prune both CI(X) and CI(Y) using CI(X) meet CI(Y)

Three implementations of constraint solving over modular integers (in progress)

- MAXC (INRIA):
  - Developed for EUCLIDE, a plateform for verifying critical C programs
  - In SICStus Prolog (700loc) + C (300loc)
  - Direct implem. Of Clockwise Intervals over 1, 2, 3, 4, 8, 16, 32 bits only
  - unsigned only, no conversions, few arithmetic and relations

- JSOLVER (ILOG)
  - Static analysis of rule-based programs (ILOG Rules)
  - Domain and Bound-consistencies on ideal integer arithmetic and
  - use of a cast function to map the results on wrap-around

- COLIBRI (CEA):
  - Constraints library used by CEA test generation tools (GATeL for LUSTRE models, PathCrawler for C code, Osmose for binary code)
  - Integer/Real/Floating points interval arithmetics (union of disjoint intervals), Congruences, Difference constraints
  - signed and unsigned cases

COLIBRI (CEA): 2 extra ideas

- For each *op* in {+,-,*,div,rem}, COLIBRI provides a modular version op_{2^n}, modular constraint propagators are handled by non modular operations:

  $$A\ op_{2^n}\ B = C \Leftrightarrow A\ op\ B = C + K * 2^n$$

  The range of K varies according to signed/unsigned, *n* and *op*.

  Example: $A +_{2^n} B = C$
  - Signed : [A,B,C] :: $[-2^{n-1}..2^{n-1}-1]$, K :: [-1..1]
  - Unsigned : [A,B,C] :: $[0..2^n-1]$, K :: [0..1]

- For each op_{2^n}, an extra argument UO :: [-1..1] allows to read / provoke an underflow (UO = -1), overflow (UO = 1) or a nominal behavior (UO = 0)

  An extra constraint maintains the invariant sign(UO) = sign(K)
  When UO = K = 0, $A\ op_{2^n}\ B = A\ op\ B$

Example: n = 3, A,B,C unsigned, A :: [2..4], B :: [5..7], C :: [0..7], UO :: [0..1]

$A +_{2^3} B =_{UO} C \rightarrow$ A + B = C + K*8 with K :: [0..1] and sign(K) = sign(UO)
$\rightarrow$ C :: [0..3, 7]

ILOG JSolver: A CP library in Java for Rule Program Analysis

For any arithmetic operator, compute intervals of Z and then project them on computer intervals using a cast function

- Let [a, b] be an interval of Z and u,v represent a, b in a (m, M) computer integer
  - $a = u + k_u(M-m+1), \quad m \leq u \leq M$
  - $b = v + k_v(M-m+1), \quad m \leq v \leq M$
- $cast_{m,M}([a, b]) = \begin{cases} [u, v] \text{ if } k_u = k_v \\ [m, M] \text{ otherwise} \end{cases}$

Further work

- Finding optimal bounds for non-linear constraints is hard

  → practical solution: relaxing optimality using over-approximations,
  e.g., X in a..b, Y in c..d then `Z = X*Y` in min(a*Y, X*c)..max(b*Y,X*d)

- Finishing our three implementations and performing a serious experimental evaluation is indispensable → next step

- Deal with constraints where distinct basis are considered,
  e.g.,

```
short x ;
long y  ;
x = (short) y ;
```