Optimising Quantified Expressions in Constraint Models

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- powerful means to compactly represent a set of expressions
- same structure in all constraint modelling languages
  - **restriction**: no decision variables in $i_1, \ldots, i_m$ and int(lb..ub)

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  automatically **improve** poorly formulated quantified expressions
- **Our Contributions**:
  - we consider **2 kinds of redundancies**
  - we propose means to **detect** and **address** those redundancies

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- **Question:** which representation is better?

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  3. $m\mathbf{A} + \sum_I E_I \equiv \sum_I \mathbf{A} + E_I$ where $m = |I|$
  4. $\mathbf{A} \vee (\forall_I E_I)) \equiv \forall_I \mathbf{A} \vee E_I$
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  5. etc
- Intuitively, we expect the outside-representation to be better... is this true for all cases?

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  - n-ary sum ($\sum$)

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- We assume the solver provides:
  - (reifyable) n-ary conjunction ($\forall$)
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  - n-ary sum ($\sum$)
- Let's look at one case (see paper for other cases):
  $\mathbf{A} \Rightarrow (\forall_I E_I) \quad \equiv \quad \forall_I \mathbf{A} \Rightarrow E_I$

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	0 *auxiliary variables* k *constraints*	1 *auxiliary variable* 2 *constraints*

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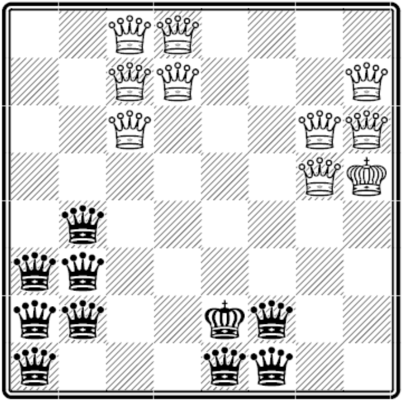
- Let's compare the representations in an example!

Example: Peaceful Army of Queens

Place two equally-sized armies of queens on a chess board such that they do not attack another, maximising the army size



Peaceful Army of Queens: Outside Representation

Non-attacking Constraints in model based on Smith et al (2004):

forall *fields(i,j) on the chess board.*

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     $\wedge$ *no black queen at field(k,j) (same row)*

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white queen at field(i,j) $\Rightarrow$
forall *k*.
no black queen at field(i,k) (same column)
$\wedge$ *no black queen at field(k,j) (same row)*
$\wedge$ *no black queen at field(i+k,j+k) (NW-diagonal)*
$\wedge$ *no black queen at field(i-k,j+k) (SW-diagonal)*
$\wedge$ *no black queen at field(i+k,j-k) (NE-diagonal)*
$\wedge$ *no black queen at field(i-k,j-k) (SE-diagonal)*

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no black queen at field(i,k) (column)
$\wedge$ **forall** k.
white queen at field(i,j) $\Rightarrow$
$\wedge$ *no black queen at field(k,j) (row)*
$\wedge$ **forall** k.
white queen at field(i,j) $\Rightarrow$
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2. We translated both models to solvers Gecode and Minion (using Tailor), generating:
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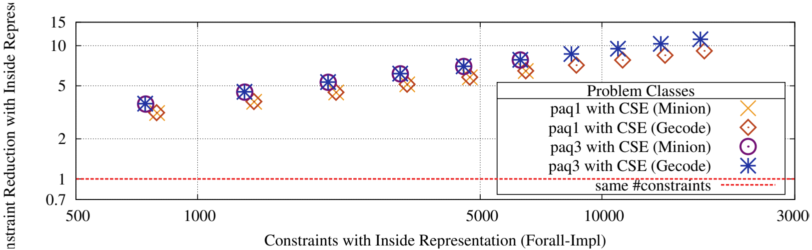
   for both models
3. We solved both representations using the same solving setup

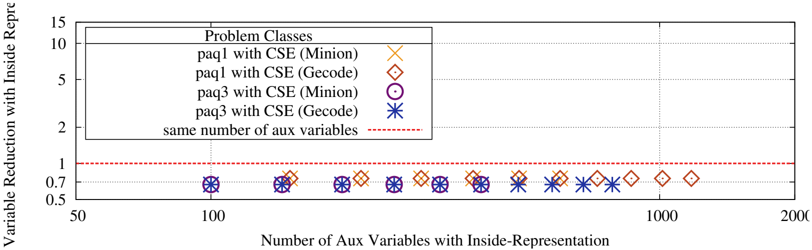
Comparing Number of Constraints

Inside-Representation has far **more** constraints than **Outside**-Representation

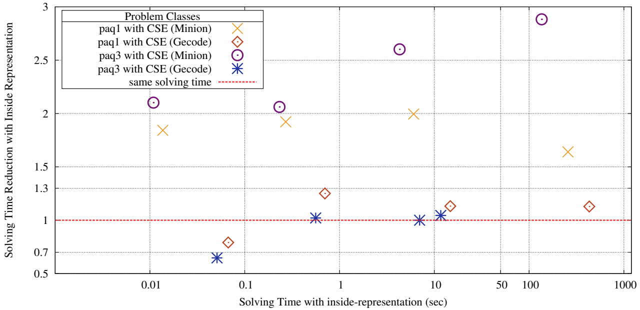


Comparing Number of Auxiliary Variables

Inside-Representation has **30% less** auxiliary variables than **Outside**-Representation



Comparing Number Solving Performance



- **Inside**-Rep. better in Minion (speedup of max. 300%)
- **Inside**-Rep. slightly better in Gecode (speedup of max. 30%)

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Conclusion on Loop-Invariant Expressions

- **Against our expectations**: it can be beneficial to move loop-invariant expressions into quantifications
- Difficult to make a **general** statement
  - depends on solver (provided propagators, architecture, etc)
  - depends on problem structure
- Tailor can **automatically reformulate** quantifications to inside/outside-representation
  - user can choose preferable representation (for each case) in translation settings

Weak Guards

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- **Example:**

forall i, j in $(1..n)$.
$(i \neq j) \quad \Rightarrow \quad \text{queen[i]} + \text{i} \neq \text{queen[j]} + \text{j}$

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  $(i \neq j) \quad \Rightarrow \quad$ queen[i] + i $\neq$ queen[j] + j
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- **Option2**: strengthen the guard!

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- **Unification Example**:
  - What is the unifier for '$x + i$' and '$x + 3$'?
  - $u = \{3/i\}$ (i substituted with 3)
- We want to demonstrate the algorithm on an example...

Strengthening the Guard in Golomb Ruler

A Golomb Ruler has *n* **ticks** such that the **distance** between each tick is **different**, minimising the length of the ruler.

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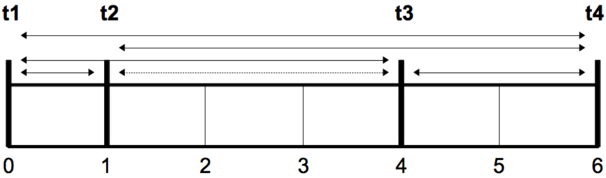
Sample Golomb Ruler with 4 ticks and length 6:



Strengthening the Guard in Golomb Ruler

‘The distances between all ticks are different’-Constraint:

Strengthening the Guard in Golomb Ruler

'The distances between all ticks are different'-Constraint:

forall $i1, i2, i3, i4$: TICKS.
$((i1>i2) \wedge (i3>i4) \wedge (i2 \neq i4)) \Rightarrow$
$(ruler[i1]-ruler[i2] \neq ruler[i3]-ruler[i4])$

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forall $i1, i2, i3, i4$: TICKS.
  ((i1>i2) $\wedge$ (i3>i4) $\wedge$ (i2$\neq$i4)) $\Rightarrow$
    (ruler[i1]-ruler[i2] $\neq$ ruler[i3]-ruler[i4])

Strengthening the Guard in Golomb Ruler

STRENGTHEN_GUARD($\forall_I : D.B_I \Rightarrow E_I$)

- (2) Compute the set of unifiers U for the two children of E_I, e_1 and e_2.

UNIFY (ruler[i1]-ruler[i2], ruler[i3]-ruler[i4]):

$$u_1 = \{i_1/i_3 \wedge i_2/i_4\} \qquad u_2 = \{i_3/i_1 \wedge i_4/i_2\}$$
$$u_3 = \{i_3/i_1 \wedge i_2/i_4\} \qquad u_4 = \{i_1/i_3 \wedge i_4/i_2\}$$

Strengthening the Guard in Golomb Ruler

STRENGTHEN_GUARD($\forall_I : D.B_I \Rightarrow E_I$)

- (3) Search U for unifiers from which we can deduce equivalence of the quantifying variables.

UNIFY (ruler[i1]-ruler[i2], ruler[i3]-ruler[i4]):

$$u_1 = \{i_1/i_3 \wedge i_2/i_4\} \qquad u_2 = \{i_3/i_1 \wedge i_4/i_2\}$$
$$u_3 = \{i_3/i_1 \wedge i_2/i_4\} \qquad u_4 = \{i_1/i_3 \wedge i_4/i_2\}$$

we deduce that $(i_1 = i_3) \wedge (i_2 = i_4)$

Strengthening the Guard in Golomb Ruler

STRENGTHEN_GUARD($\forall_I : D.B_I \Rightarrow E_I$)

- (4) Add lex-ordering constraint C on all quantifying variables whose equivalence renders e_1 and e_2 equivalent

C: $i_1, i_2 \leq_{lex} i_3, i_4$
hence $(i_1 \leq i_3) \wedge (i_1 < i_3 \vee i_2 \leq i_4)$

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Yielding the constraint with **strengthend guard:**

forall $i1, i2, i3, i4$: TICKS.
((i1>i2) $\wedge$ (i3>i4) $\wedge$ (i2$\neq$i4) $\wedge$
$(i_1 \leq i_3) \wedge (i_1 < i_3 \vee i_2 \leq i_4))$
$\Rightarrow$
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Strengthening the Guard in Golomb Ruler

Yielding the constraint with **strengthend guard:**

forall $i1, i2, i3, i4$: TICKS.
((i1>i2) ∧ (i3>i4) ∧ (i2≠i4) ∧
$(i_1 \leq i_3) \wedge (i_1 < i_3 \vee i_2 \leq i_4))$
⇒
(ruler[i1]-ruler[i2] ≠ ruler[i3]-ruler[i4])

However: we have not implemented the algorithm yet!

Effects of Duplicate constraints

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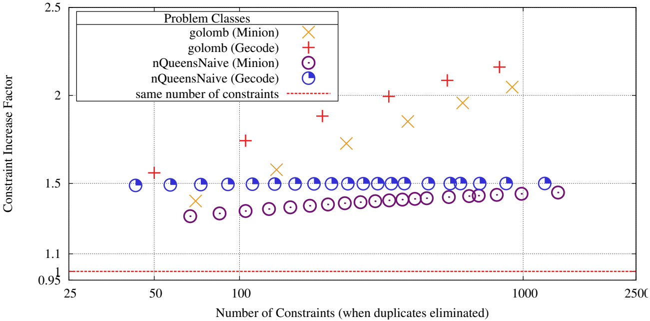
Effects of Duplicate constraints

- How bad is the **effect of duplicate constraints** due to weak guards?
  - in other words: is it worth putting energy into strengthening guards?
- We analyse the effects on two naive models in solver Minion and Gecode:
  - Naive n-Queens
  - Naive Golomb Ruler

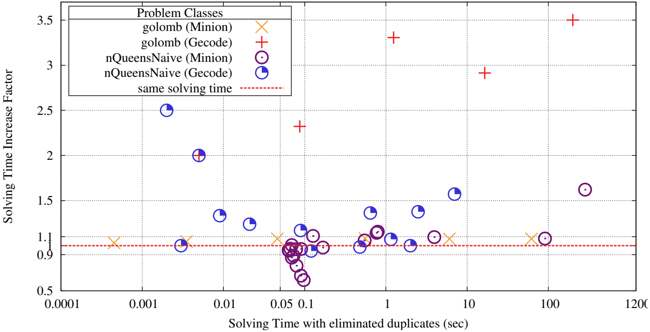
The Number of Duplicate Constraints

For both solvers: constant for n-Queens, linear within Golomb Ruler



Effect on Solving Performance

strong effect in Gecode, mild effect in Minion



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- We have an idea on how to **strengthen guards** to address this redundancy
- We still need to implement/test/refine the algorithm..

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- We can already provide some enhancement
- But there is still a lot to investigate!

Thank You.