Introduction to Constraint Programming

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Licence

Acknowledgments

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`http://4c.ucc.ie/~hsimonis/ELearning/index.htm`
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Objectives

- Overview of Core Constraint Programming
- Three Main Concepts
  - Constraint Propagation
  - Global Constraints
  - Customizing Search
- Based on Examples, not Formal Description

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Outline

- Why Constraint Programming?
- Constraint Propagation
- Global Constraints
- Customizing Search
- What is missing?

Examples in ECLiPSe

- Open sourced constraint programming language
- Development goes back to 1985
- ECRC, ICL, IC-Parc, PTL, Cisco
- `http://www.eclipse-clp.org/`
- Specialities
  - Develop new solvers for specific domains
  - Integration with MIP

ECLiPSe ELearning Course

- Self-study course in constraint programming
- Supported by Cisco Systems and Silicon Valley Community Foundation
- Multi-media format, video lectures, slides, handout etc
- `http://4c.ucc.ie/~hsimonis/ELearning/index.htm`

Constraint Programming - in a nutshell

- Declarative description of problems with
  - *Variables* which range over (finite) sets of values
  - *Constraints* over subsets of variables which restrict possible value combinations
  - A *solution* is a value assignment which satisfies all constraints
- Constraint propagation/reasoning
  - Removing inconsistent values for variables
  - Detect failure if constraint can not be satisfied
  - Interaction of constraints via shared variables
  - Incomplete
- Search
  - User controlled assignment of values to variables
  - Each step triggers constraint propagation
- Different domains require/allow different methods

Basic Process

Problem

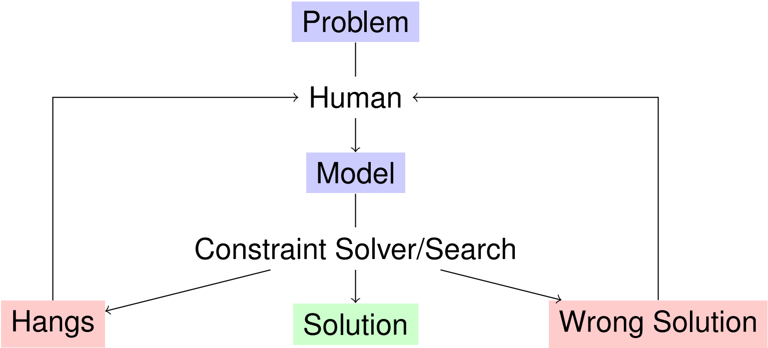
↓ Human

Model

↓ Constraint Solver/Search

Solution

More Realistic



Dual Role of Model

- Allows Human to Express Problem
  - Close to Problem Domain
  - Constraints as Abstractions
- Allows Solver to Execute
  - Variables as Communication Mechanism
  - Constraints as Algorithms

Part I

Basic Constraint Propagation

Example 1: SEND+MORE=MONEY

- Example of Finite Domain Constraint Problem
- Models and Programs
- Constraint Propagation and Search
- Some Basic Constraints: linear arithmetic, alldifferent, disequality
- A Built-in search
- Visualizers for variables, constraints and search

Outline

Problem Definition

A Crypt-Arithmetic Puzzle

We begin with the definition of the SEND+MORE=MONEY puzzle. It is often shown in the form of a hand-written addition:

	S	E	N	D
+	M	O	R	E
M	O	N	E	Y

Rules

- Each character stands for a digit from 0 to 9.
- Numbers are built from digits in the usual, positional notation.
- Repeated occurrence of the same character denote the same digit.
- Different characters denote different digits.
- Numbers do not start with a zero.
- The equation must hold.

		S	E	N	D
+		M	O	R	E
	M	O	N	E	Y

Outline

Model

- Each character is a variable, which ranges over the values 0 to 9.
- An *alldifferent* constraint between all variables, which states that two different variables must have different values. This is a very common constraint, which we will encounter in many other problems later on.
- Two *disequality constraints* (variable X must be different from value V) stating that the variables at the beginning of a number can not take the value 0.
- An arithmetic *equality constraint* linking all variables with the proper coefficients and stating that the equation must hold.

Program Sendmory

```
:- module(sendmory). ⇨ Define Module
:- export(sendmory/1).
:- lib(ic).
sendmory(L):-
      L = [S,E,N,D,M,O,R,Y],
      L :: 0..9,
      alldifferent(L),
      S #\= 0, M #\= 0,
      1000*S + 100*E + 10*N + D +
      1000*M + 100*O + 10*R + E #=
      10000*M + 1000*O + 100*N + 10*E + Y,
      labeling(L).
```

Program Sendmory

```
:- module(sendmory).
:- export(sendmory/1).  ➪ Make predicate visible
:- lib(ic).
sendmory(L):-
        L = [S,E,N,D,M,O,R,Y],
        L :: 0..9,
        alldifferent(L),
        S #\= 0, M #\= 0,
        1000*S + 100*E + 10*N + D +
        1000*M + 100*O + 10*R + E #=
        10000*M + 1000*O + 100*N + 10*E + Y,
        labeling(L).
```

Program Sendmory

```
:- module(sendmory).
:- export(sendmory/1).
:- lib(ic).  ⇨ Use ic library
sendmory(L):-
        L = [S,E,N,D,M,O,R,Y],
        L :: 0..9,
        alldifferent(L),
        S #\= 0, M #\= 0,
        1000*S + 100*E + 10*N + D +
        1000*M + 100*O + 10*R + E #=
        10000*M + 1000*O + 100*N + 10*E + Y,
        labeling(L).
```

Program Sendmory

```
:- module(sendmory).
:- export(sendmory/1).
:- lib(ic).
sendmory(L):-  Predicate definition
    L = [S,E,N,D,M,O,R,Y],
    L :: 0..9,
    alldifferent(L),
    S #\= 0, M #\= 0,
    1000*S + 100*E + 10*N + D +
    1000*M + 100*O + 10*R + E #=
    10000*M + 1000*O + 100*N + 10*E + Y,
    labeling(L).
```

Program Sendmory

```
:- module(sendmory).
:- export(sendmory/1).
:- lib(ic).
sendmory(L):-
        L = [S,E,N,D,M,O,R,Y], ⇨ Define list
        L :: 0..9,
        alldifferent(L),
        S #\= 0, M #\= 0,
        1000*S + 100*E + 10*N + D +
        1000*M + 100*O + 10*R + E #=
        10000*M + 1000*O + 100*N + 10*E + Y,
        labeling(L).
```

Program Sendmory

```
:- module(sendmory).
:- export(sendmory/1).
:- lib(ic).
sendmory(L):-
      L = [S,E,N,D,M,O,R,Y],
      L :: 0..9,⇨ Define integer domain 0..9
      alldifferent(L),
      S #\= 0, M #\= 0,
      1000*S + 100*E + 10*N + D +
      1000*M + 100*O + 10*R + E #=
      10000*M + 1000*O + 100*N + 10*E + Y,
      labeling(L).
```

Program Sendmory

```
:- module(sendmory).
:- export(sendmory/1).
:- lib(ic).
sendmory(L):-
     L = [S,E,N,D,M,O,R,Y],
     L :: 0..9,
     alldifferent(L), ⇨ Digits must be different
     S #\= 0, M #\= 0,
     1000*S + 100*E + 10*N + D +
     1000*M + 100*O + 10*R + E #=
     10000*M + 1000*O + 100*N + 10*E + Y,
     labeling(L).
```

Program Sendmory

```
:- module(sendmory).
:- export(sendmory/1).
:- lib(ic).
sendmory(L):-
      L = [S,E,N,D,M,O,R,Y],
      L :: 0..9,
      alldifferent(L),
      S #\= 0, M #\= 0, ⇨ Numbers don't start with 0
      1000*S + 100*E + 10*N + D +
      1000*M + 100*O + 10*R + E #=
      10000*M + 1000*O + 100*N + 10*E + Y,
      labeling(L).
```

```
      S  E  N  D
  +   M  O  R  E
  ---------------
   M  O  N  E  Y
```

Program Sendmory

```
:- module(sendmory).
:- export(sendmory/1).
:- lib(ic).
sendmory(L):-
     L = [S,E,N,D,M,O,R,Y],
     L :: 0..9,
     alldifferent(L),
     S #\= 0, M #\= 0,
     1000*S + 100*E + 10*N + D +
     1000*M + 100*O + 10*R + E #=
     10000*M + 1000*O + 100*N + 10*E + Y,
     labeling(L).
```

		S	E	N	D
+		M	O	R	E
	M	O	N	E	Y

Program Sendmory

```
:- module(sendmory).
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sendmory(L):-
     L = [S,E,N,D,M,O,R,Y],
     L :: 0..9,
     alldifferent(L),
     S #\= 0, M #\= 0,
     1000*S + 100*E + 10*N + D +
     1000*M + 100*O + 10*R + E #=
     10000*M + 1000*O + 100*N + 10*E + Y,
     labeling(L).
```

⇨ *built-in search routine*

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Program Sendmory

```
:- module(sendmory).
:- export(sendmory/1).
:- lib(ic).
sendmory(L):-
      L = [S,E,N,D,M,O,R,Y],
      L :: 0..9,
      alldifferent(L),
      S #\= 0, M #\= 0,
      1000*S + 100*E + 10*N + D +
      1000*M + 100*O + 10*R + E #=
      10000*M + 1000*O + 100*N + 10*E + Y,
      labeling(L).
```

General Program Structure

```
:- module(sendmory).
:- export(sendmory/1).
:- lib(ic).
sendmory(L):-
        L = [S,E,N,D,M,O,R,Y],⇨ Variables
        L :: 0..9,
        alldifferent(L),
        S #\= 0, M #\= 0,
        1000*S + 100*E + 10*N + D +
        1000*M + 100*O + 10*R + E #=
        10000*M + 1000*O + 100*N + 10*E + Y,
        labeling(L).
```

General Program Structure

```
:- module(sendmory).
:- export(sendmory/1).
:- lib(ic).
sendmory(L):-
      L = [S,E,N,D,M,O,R,Y],
      L :: 0..9,
      alldifferent(L), ⇨ Constraints
      S #\= 0, M #\= 0,
      1000*S + 100*E + 10*N + D +
      1000*M + 100*O + 10*R + E #=
      10000*M + 1000*O + 100*N + 10*E + Y,
      labeling(L).
```

General Program Structure

```
:- module(sendmory).
:- export(sendmory/1).
:- lib(ic).
sendmory(L):-
        L = [S,E,N,D,M,O,R,Y],
        L :: 0..9,
        alldifferent(L),
        S #\= 0, M #\= 0,
        1000*S + 100*E + 10*N + D +
        1000*M + 100*O + 10*R + E #=
        10000*M + 1000*O + 100*N + 10*E + Y,
        labeling(L). ⇨ Search
```

Choice of Model

- This is *one* model, not *the* model of the problem
- Many possible alternatives
- Choice often depends on your constraint system
  - Constraints available
  - Reasoning attached to constraints
- Not always clear which is the *best* model
- Often: Not clear what is the *problem*

Running the program

- To run the program, we have to enter the query
  - `sendmory:sendmory(L).`
- Result
  - `L = [9, 5, 6, 7, 1, 0, 8, 2]`
  - `yes (0.00s cpu, solution 1, maybe more)`

Question

- But how did the program come up with this solution?

Outline

1. Problem
2. Program
3. Constraint Setup
   - Domain Definition
   - Alldifferent Constraint
   - Disequality Constraints
   - Equality Constraint
4. Search
5. Points to Remember

Domain Definition

```
L = [S,E,N,D,M,O,R,Y],
L :: 0..9,
```

$$[S, E, N, D, M, O, R, Y] \in \{0..9\}$$

Domain Visualization

	0	1	2	3	4	5	6	7	8	9
S										
E										
N										
D										
M										
O										
R										
Y										

Domain Visualization

Rows = Variables

	0	1	2	3	4	5	6	7	8	9
S										
E										
N										
D										
M										
O										
R										
Y										

Domain Visualization

Columns = Values

	0	1	2	3	4	5	6	7	8	9
S										
E										
N										
D										
M										
O										
R										
Y										

Domain Visualization

	0	1	2	3	4	5	6	7	8	9
S										
E										
N										
D										
M			Cells=	State						
O										
R										
Y										

Alldifferent Constraint

```
alldifferent(L),
```

- Built-in of `ic` library
- No initial propagation possible
- *Suspends*, waits until variables are changed
- When variable is fixed, remove value from domain of other variables
- *Forward checking*

Alldifferent Visualization

Uses the same representation as the domain visualizer

	0	1	2	3	4	5	6	7	8	9
S										
E										
N										
D										
M										
O										
R										
Y										

Disequality Constraints

```
S #\= 0, M#\= 0,
```

Remove value from domain

$$S \in \{1..9\}, M \in \{1..9\}$$

Constraints solved, can be removed

Domains after Disequality

	0	1	2	3	4	5	6	7	8	9
S										
E										
N										
D										
M										
O										
R										
Y										

Equality Constraint

- Normalization of linear terms
  - Single occurence of variable
  - Positive coefficients
- Propagation

Normalization

	1000*S+	100*E+	10*N+	D
	+1000*M+	100*O+	10*R+	E
10000*M+	1000*O+	100*N+	10*E+	Y

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Normalization

	1000*S+	100*E+	10*N+	D
	+**1000*M**+	100*O+	10*R+	E
10000*M+	1000*O+	100*N+	10*E+	Y

Normalization

	1000*S+	100*E+	10*N+	D
	+	100*O+	10*R+	E
9000*M+	1000*O+	100*N+	10*E+	Y

Normalization

	1000*S+	100*E+	10*N+	D
	+	**100*O**+	10*R+	E
9000*M+	**1000*O**+	100*N+	10*E+	Y

Normalization

	1000*S+	100*E+	10*N+	D
		+	10*R+	E
9000*M+	**900*O**+	100*N+	10*E+	Y

Normalization

	1000*S+	100*E+	**10*N**+	D
		+	10*R+	E
9000*M+	900*O+	**100*N**+	10*E+	Y

Normalization

	1000*S+	100*E+		D
		+	10*R+	E
9000*M+	900*O+	**90*N**+	10*E+	Y

Normalization

	1000*S+	**100*E+**		D
		+	10*R+	**E**
9000*M+	900*O+	90*N+	**10*E+**	Y

Normalization

	1000*S+	91*E+		D
		+	10*R	
9000*M+	900*O+	90*N+		Y

Simplified Equation

$$1000 * S + 91 * E + 10 * R + D = 9000 * M + 900 * O + 90 * N + Y$$

Propagation

$$1000 * S^{1..9} + 91 * E^{0..9} + 10 * R^{0..9} + D^{0..9} = 9000 * M^{1..9} + 900 * O^{0..9} + 90 * N^{0..9} + Y^{0..9}$$

Propagation

$$\underbrace{1000 * S^{1..9} + 91 * E^{0..9} + 10 * R^{0..9} + D^{0..9}}_{1000..9918} =$$

$$\underbrace{9000 * M^{1..9} + 900 * O^{0..9} + 90 * N^{0..9} + Y^{0..9}}_{9000..89919}$$

Propagation

$$\underbrace{1000 * S^{1..9} + 91 * E^{0..9} + 10 * R^{0..9} + D^{0..9}}_{9000..9918} =$$

$$\underbrace{9000 * M^{1..9} + 900 * O^{0..9} + 90 * N^{0..9} + Y^{0..9}}_{9000..9918}$$

Propagation

$$\underbrace{1000 * S^{1..9} + 91 * E^{0..9} + 10 * R^{0..9} + D^{0..9}}_{9000..9918} =$$

$$\underbrace{9000 * M^{1..9} + 900 * O^{0..9} + 90 * N^{0..9} + Y^{0..9}}_{9000..9918}$$

Deduction:

$$M = 1, S = 9, O \in \{0..1\}$$

Propagation

$$\underbrace{1000 * S^{1..9} + 91 * E^{0..9} + 10 * R^{0..9} + D^{0..9}}_{9000..9918} =$$

$$\underbrace{9000 * M^{1..9} + 900 * O^{0..9} + 90 * N^{0..9} + Y^{0..9}}_{9000..9918}$$

Deduction:

$$M = 1, S = 9, O \in \{0..1\}$$

Why? Skip

Consider lower bound for S

$$\underbrace{1000 * S^{1..9} + 91 * E^{0..9} + 10 * R^{0..9} + D^{0..9}}_{9000..9918} = \underbrace{9000 * M^{1..9} + 900 * O^{0..9} + 90 * N^{0..9} + Y^{0..9}}_{9000..9918}$$

- Lower bound of equation is 9000
- Rest of lhs (left hand side) $(91 * E^{0..9} + 10 * R^{0..9} + D^{0..9})$ is atmost 918
- S must be greater or equal to $\frac{9000-918}{1000} = 8.082$
  - otherwise lower bound of equation not reached by lhs
- S is integer, therefore $S \geq \lceil \frac{9000-918}{1000} \rceil = 9$
- S has upper bound of 9, so $S = 9$

Consider upper bound of M

$$\underbrace{1000 * S^{1..9} + 91 * E^{0..9} + 10 * R^{0..9} + D^{0..9}}_{9000..9918} = \underbrace{9000 * M^{1..9} + 900 * O^{0..9} + 90 * N^{0..9} + Y^{0..9}}_{9000..9918}$$

- Upper bound of equation is 9918
- Rest of rhs (right hand side) $900 * O^{0..9} + 90 * N^{0..9} + Y^{0..9}$ is at least 0
- M must be smaller or equal to $\frac{9918-0}{9000} = 1.102$
- M must be integer, therefore $M \leq \lfloor \frac{9918-0}{9000} \rfloor = 1$
- M has lower bound of 1, so $M = 1$

Consider upper bound of O

$$\underbrace{1000 * S^{1..9} + 91 * E^{0..9} + 10 * R^{0..9} + D^{0..9}}_{9000..9918} = \underbrace{9000 * M^{1..9} + 900 * O^{0..9} + 90 * N^{0..9} + Y^{0..9}}_{9000..9918}$$

- Upper bound of equation is 9918
- Rest of rhs (right hand side) $9000 * 1 + 90 * N^{0..9} + Y^{0..9}$ is at least 9000
- O must be smaller or equal to $\frac{9918-9000}{900} = 1.02$
- O must be integer, therefore $O \leq \lfloor \frac{9918-9000}{900} \rfloor = 1$
- O has lower bound of 0, so $O \in \{0..1\}$

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Propagation of equality: Result

	0	1	2	3	4	5	6	7	8	9
S		-	-	-	-	-	-	-	-	✹
E										
N										
D										
M		✹	-	-	-	-	-	-	-	-
O			✖	✖	✖	✖	✖	✖	✖	✖
R										
Y										

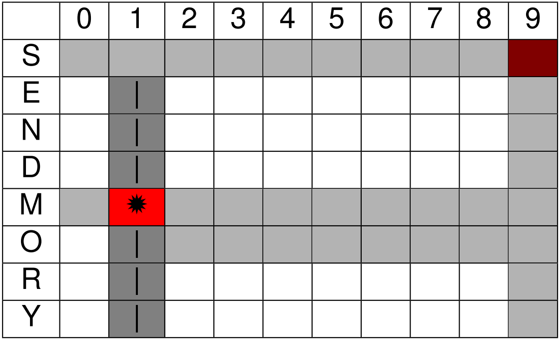
Propagation of alldifferent

	0	1	2	3	4	5	6	7	8	9
S		-	-	-	-	-	-	-	-	✹
E										
N										
D										
M		✹	-	-	-	-	-	-	-	-
O			✖	✖	✖	✖	✖	✖	✖	✖
R										
Y										

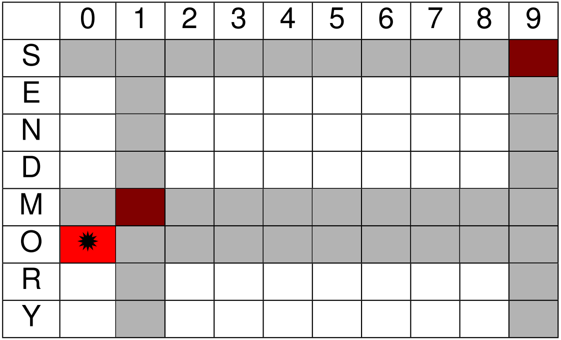
Propagation of alldifferent

	0	1	2	3	4	5	6	7	8	9
S										✸
E										∣
N										∣
D										∣
M		✸								
O										
R										∣
Y										∣

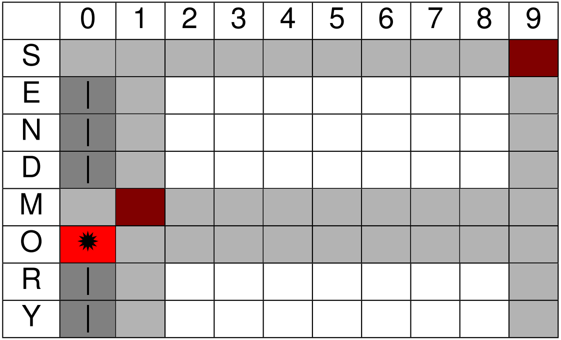
Propagation of alldifferent



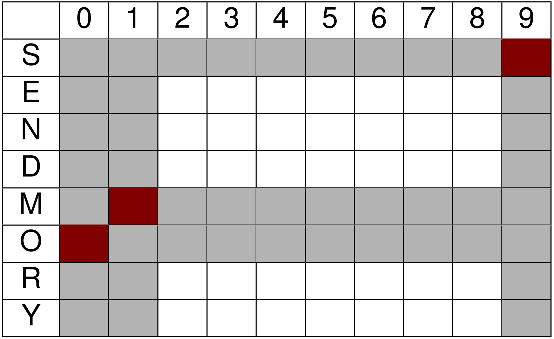
Propagation of alldifferent



Propagation of alldifferent



Propagation of alldifferent



$$O = 0, [E, R, D, N, Y] \in \{2..8\}$$

Waking the equality constraint

- Triggered by assignment of variables
- *or* update of lower or upper bound

Removal of constants

$$1000 * 9 + 91 * E^{2..8} + 10 * R^{2..8} + D^{2..8} =$$
$$9000 * 1 + 900 * 0 + 90 * N^{2..8} + Y^{2..8}$$

Removal of constants

$$\mathbf{1000} * \mathbf{9} + 91 * E^{2..8} + 10 * R^{2..8} + D^{2..8} =$$
$$\mathbf{9000} * \mathbf{1} + \mathbf{900} * \mathbf{0} + 90 * N^{2..8} + Y^{2..8}$$

Removal of constants

$$91 * E^{2..8} + 10 * R^{2..8} + D^{2..8} = 90 * N^{2..8} + Y^{2..8}$$

Propagation of equality (Iteration 1)

$$\underbrace{91 * E^{2..8} + 10 * R^{2..8} + D^{2..8}}_{204..816} = \underbrace{90 * N^{2..8} + Y^{2..8}}_{182..728}$$

Propagation of equality (Iteration 1)

$$\underbrace{91 * E^{2..8} + 10 * R^{2..8} + D^{2..8} = 90 * N^{2..8} + Y^{2..8}}_{204..728}$$

Propagation of equality (Iteration 1)

$$\underbrace{91 * E^{2..8} + 10 * R^{2..8} + D^{2..8} = 90 * N^{2..8} + Y^{2..8}}_{204..728}$$

$$N \geq 3 = \lceil \frac{204 - 8}{90} \rceil, E \leq 7 = \lfloor \frac{728 - 22}{91} \rfloor$$

Propagation of equality (Iteration 2)

$$91 * E^{2..7} + 10 * R^{2..8} + D^{2..8} = 90 * N^{3..8} + Y^{2..8}$$

Propagation of equality (Iteration 2)

$$\underbrace{91 * E^{2..7} + 10 * R^{2..8} + D^{2..8}}_{204..725} = \underbrace{90 * N^{3..8} + Y^{2..8}}_{272..728}$$

Propagation of equality (Iteration 2)

$$\underbrace{91 * E^{2..7} + 10 * R^{2..8} + D^{2..8} = 90 * N^{3..8} + Y^{2..8}}_{272..725}$$

Propagation of equality (Iteration 2)

$$\underbrace{91 * E^{2..7} + 10 * R^{2..8} + D^{2..8} = 90 * N^{3..8} + Y^{2..8}}_{272..725}$$

$$E \geq 3 = \lceil \frac{272 - 88}{91} \rceil$$

Propagation of equality (Iteration 3)

$$91 * E^{3..7} + 10 * R^{2..8} + D^{2..8} = 90 * N^{3..8} + Y^{2..8}$$

Propagation of equality (Iteration 3)

$$\underbrace{91 * E^{3..7} + 10 * R^{2..8} + D^{2..8}}_{295..725} = \underbrace{90 * N^{3..8} + Y^{2..8}}_{272..728}$$

Propagation of equality (Iteration 3)

$$\underbrace{91 * E^{3..7} + 10 * R^{2..8} + D^{2..8} = 90 * N^{3..8} + Y^{2..8}}_{295..725}$$

Propagation of equality (Iteration 3)

$$\underbrace{91 * E^{3..7} + 10 * R^{2..8} + D^{2..8} = 90 * N^{3..8} + Y^{2..8}}_{295..725}$$

$$N \geq 4 = \lceil \frac{295 - 8}{90} \rceil$$

Propagation of equality (Iteration 4)

$$91 * E^{3..7} + 10 * R^{2..8} + D^{2..8} = 90 * N^{4..8} + Y^{2..8}$$

Propagation of equality (Iteration 4)

$$\underbrace{91 * E^{3..7} + 10 * R^{2..8} + D^{2..8}}_{295..725} = \underbrace{90 * N^{4..8} + Y^{2..8}}_{362..728}$$

Propagation of equality (Iteration 4)

$$\underbrace{91 * E^{3..7} + 10 * R^{2..8} + D^{2..8} = 90 * N^{4..8} + Y^{2..8}}_{362..725}$$

Propagation of equality (Iteration 4)

$$\underbrace{91 * E^{3..7} + 10 * R^{2..8} + D^{2..8} = 90 * N^{4..8} + Y^{2..8}}_{362..725}$$

$$E \geq 4 = \lceil \frac{362 - 88}{91} \rceil$$

Propagation of equality (Iteration 5)

$$91 * E^{4..7} + 10 * R^{2..8} + D^{2..8} = 90 * N^{4..8} + Y^{2..8}$$

Propagation of equality (Iteration 5)

$$\underbrace{91 * E^{4..7} + 10 * R^{2..8} + D^{2..8}}_{386..725} = \underbrace{90 * N^{4..8} + Y^{2..8}}_{362..728}$$

Propagation of equality (Iteration 5)

$$\underbrace{91 * E^{4..7} + 10 * R^{2..8} + D^{2..8} = 90 * N^{4..8} + Y^{2..8}}_{386..725}$$

Propagation of equality (Iteration 5)

$$\underbrace{91 * E^{4..7} + 10 * R^{2..8} + D^{2..8} = 90 * N^{4..8} + Y^{2..8}}_{386..725}$$

$$N \geq 5 = \lceil \frac{386 - 8}{90} \rceil$$

Propagation of equality (Iteration 6)

$$91 * E^{4..7} + 10 * R^{2..8} + D^{2..8} = 90 * N^{5..8} + Y^{2..8}$$

Propagation of equality (Iteration 6)

$$\underbrace{91 * E^{4..7} + 10 * R^{2..8} + D^{2..8}}_{386..725} = \underbrace{90 * N^{5..8} + Y^{2..8}}_{452..728}$$

Propagation of equality (Iteration 6)

$$\underbrace{91 * E^{4..7} + 10 * R^{2..8} + D^{2..8} = 90 * N^{5..8} + Y^{2..8}}_{452..725}$$

Propagation of equality (Iteration 6)

$$\underbrace{91 * E^{4..7} + 10 * R^{2..8} + D^{2..8} = 90 * N^{5..8} + Y^{2..8}}_{452..725}$$

$$N \geq 5 = \lceil \frac{452 - 8}{90} \rceil, E \geq 4 = \lceil \frac{452 - 88}{91} \rceil$$

No further propagation at this point

Cork Constraint Computation Centre

Domains after setup

	0	1	2	3	4	5	6	7	8	9
S										
E										
N										
D										
M										
O										
R										
Y										

Outline

1. Problem
2. Program
3. Constraint Setup
4. Search
   - Step 1
   - Step 2
   - Further Steps
   - Solution
5. Points to Remember

`labeling` built-in

`labeling([S,E,N,D,M,O,R,Y])`

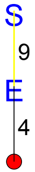
- Try variable is order given
- Try values starting from smallest value in domain
- When failing, backtrack to last open choice
- *Chronological Backtracking*
- *Depth First search*

Search Tree Step 1

Variable *S* already fixed

Step 2, Alternative $E = 4$

Variable $E \in \{4..7\}$, first value tested is 4



Assignment $E = 4$

	0	1	2	3	4	5	6	7	8	9
S										
E					✹	-	-	-		
N										
D										
M										
O										
R										
Y										

Propagation of $E = 4$, equality constraint

$$91 * 4 + 10 * R^{2..8} + D^{2..8} = 90 * N^{5..8} + Y^{2..8}$$

Propagation of $E = 4$, equality constraint

$$\underbrace{91 * 4 + 10 * R^{2..8} + D^{2..8}}_{386..452} = \underbrace{90 * N^{5..8} + Y^{2..8}}_{452..728}$$

Propagation of $E = 4$, equality constraint

$$\underbrace{91 * 4 + 10 * R^{2..8} + D^{2..8} = 90 * N^{5..8} + Y^{2..8}}_{452}$$

Propagation of $E = 4$, equality constraint

$$\underbrace{91 * 4 + 10 * R^{2..8} + D^{2..8} = 90 * N^{5..8} + Y^{2..8}}_{452}$$

$$N = 5, Y = 2, R = 8, D = 8$$

Result of equality propagation

	0	1	2	3	4	5	6	7	8	9
S										
E										
N						✹	-	-	-	
D			-	-	-	-	-	-	✹	
M										
O										
R			-	-	-	-	-	-	✹	
Y			✹	-	-	-	-	-	-	

Propagation of alldifferent

	0	1	2	3	4	5	6	7	8	9
S										
E										
N						✸	-	-	-	
D			-	-	-	-	-	-	✸	
M										
O										
R			-	-	-	-	-	-	✸	
Y			✸	-	-	-	-	-	-	

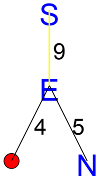
Propagation of alldifferent

	0	1	2	3	4	5	6	7	8	9
S									|	
E									|	
N						✹	-	-	|	
D			-	-	-	-	-	-	✹	
M									|	
O									|	
R			-	-	-	-	-	-	✹	
Y			✹	-	-	-	-	-	|	

Alldifferent fails!

Step 2, Alternative $E = 5$

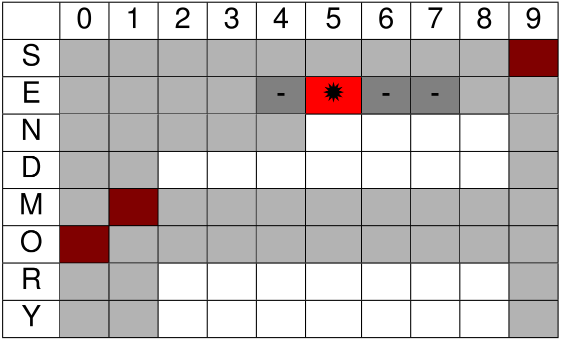
Return to last open choice, E, and test next value



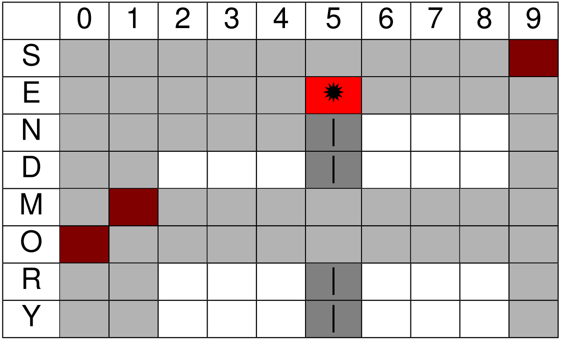
Assignment $E = 5$

	0	1	2	3	4	5	6	7	8	9
S										
E					-	✹	-	-		
N										
D										
M										
O										
R										
Y										

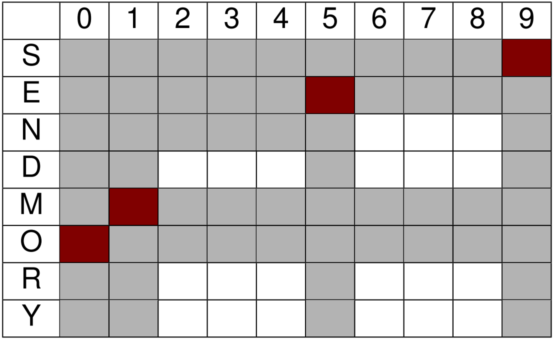
Propagation of alldifferent



Propagation of alldifferent



Propagation of alldifferent



$$N \neq 5, N \geq 6$$

Propagation of equality

$$91 * 5 + 10 * R^{2..8} + D^{2..8} = 90 * N^{6..8} + Y^{2..8}$$

Propagation of equality

$$\underbrace{91 * 5 + 10 * R^{2..8} + D^{2..8}}_{477..543} = \underbrace{90 * N^{6..8} + Y^{2..8}}_{542..728}$$

Propagation of equality

$$\underbrace{91 * 5 + 10 * R^{2..8} + D^{2..8} = 90 * N^{6..8} + Y^{2..8}}_{542..543}$$

Cork Constraint Computation Centre

Propagation of equality

$$\underbrace{91 * 5 + 10 * R^{2..8} + D^{2..8} = 90 * N^{6..8} + Y^{2..8}}_{542..543}$$

$$N = 6, Y \in \{2, 3\}, R = 8, D \in \{7..8\}$$

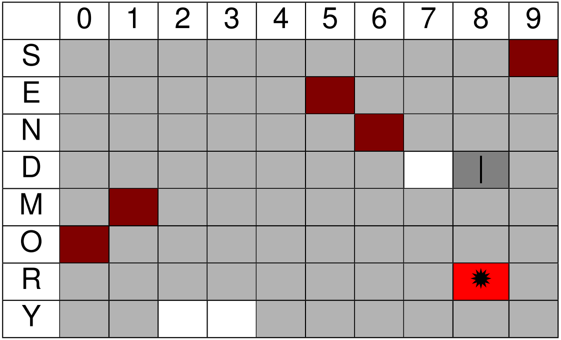
Result of equality propagation

	0	1	2	3	4	5	6	7	8	9
S										
E										
N							✹	-	-	
D			✖	✖	✖		✖			
M										
O										
R			-	-	-		-	-	✹	
Y					✖		✖	✖	✖	

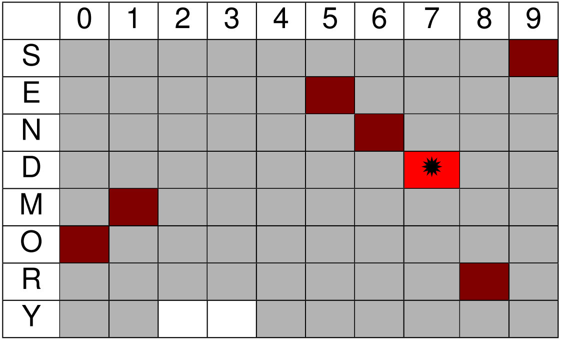
Propagation of `alldifferent`

	0	1	2	3	4	5	6	7	8	9
S										
E										
N							✹	-	-	
D			✖	✖	✖		✖			
M										
O										
R			-	-	-		-	-	✹	
Y					✖		✖	✖	✖	

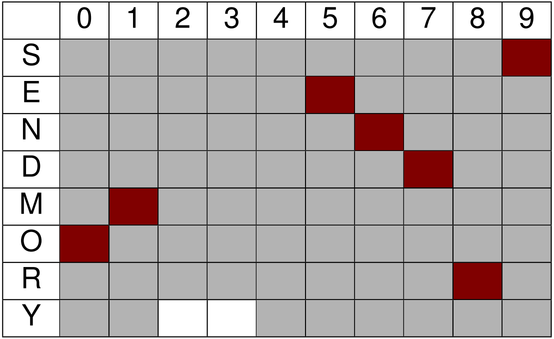
Propagation of `alldifferent`



Propagation of `alldifferent`



Propagation of `alldifferent`



$D = 7$

Propagation of equality

$$91 * 5 + 10 * 8 + 7 = 90 * 6 + Y^{2..3}$$

Propagation of equality

$$\underbrace{91 * 5 + 10 * 8 + 7}_{542} = \underbrace{90 * 6 + Y^{2..3}}_{542..543}$$

Propagation of equality

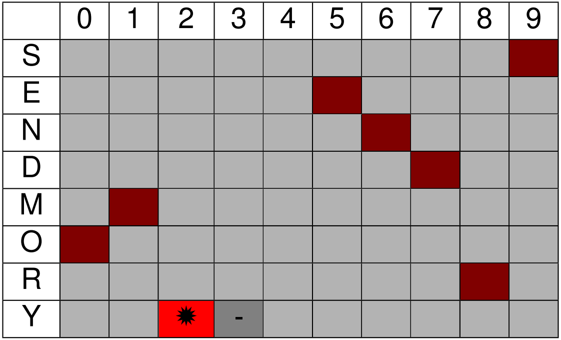
$$\underbrace{91 * 5 + 10 * 8 + 7 = 90 * 6 + Y^{2..3}}_{542}$$

Propagation of equality

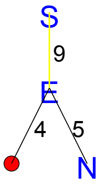
$$\underbrace{91 * 5 + 10 * 8 + 7 = 90 * 6 + Y^{2..3}}_{542}$$

$$Y = 2$$

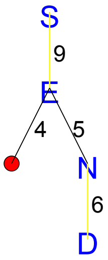
Last propagation step



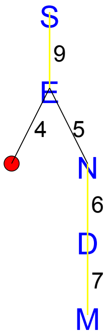
Further Steps: Nothing more to do



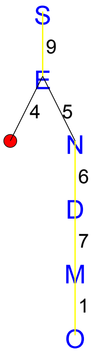
Further Steps: Nothing more to do



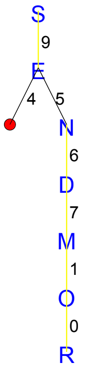
Further Steps: Nothing more to do



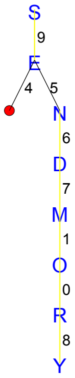
Further Steps: Nothing more to do



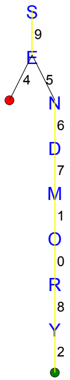
Further Steps: Nothing more to do



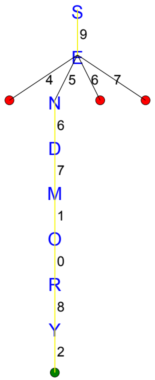
Further Steps: Nothing more to do



Further Steps: Nothing more to do



Complete Search Tree



Solution

	9	5	6	7
+	1	0	8	5
1	0	6	5	2

Outline

Points to Remember

- Constraint models are expressed by variables and constraints.
- Problems can have many different models, which can behave quite differently. Choosing the best model is an art.
- Constraints can take many different forms.
- Propagation deals with the interaction of variables and constraints.
- It removes some values that are inconsistent with a constraint from the domain of a variable.
- Constraints only communicate via shared variables.

Points to Remember

- Propagation usually is not sufficient, search may be required to find a solution.
- Propagation is data driven, and can be quite complex even for small examples.
- The default search uses chronological depth-first backtracking, systematically exploring the complete search space.
- The search choices and propagation are interleaved, after every choice some more propagation may further reduce the problem.

Part II

Global Constraints

Example 2: Sudoku

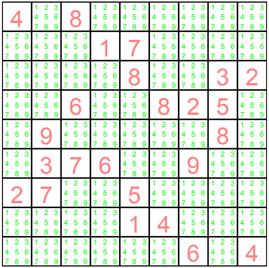
- Global Constraints
  - Powerful modelling abstractions
  - Non-trivial propagation
  - Different consistency levels
- Example: Sudoku puzzle

Outline

Problem Definition

Sudoku

Fill in numbers from 1 to 9 so that each row, column and block contain each number exactly once



Problem Definition

Sudoku

Fill in numbers from 1 to 9 so that each row, column and block contain each number exactly once

4	1 2 3 4 5 6 7 8 9	8	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1	7	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	8	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	3	2
1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	6	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	8	2	5	1 2 3 4 5 6 7 8 9
1 2 3 4 5 6 7 8 9	9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	8	1 2 3 4 5 6 7 8 9
1 2 3 4 5 6 7 8 9	3	7	6	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
2	7	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	5	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1	4	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	6	1 2 3 4 5 6 7 8 9	4

4	2	8	5	6	3	1	7	9
3	5	9	1	7	2	4	6	8
7	6	1	4	8	9	5	3	2
1	4	6	3	9	8	2	5	7
5	9	2	7	4	1	3	8	6
8	3	7	6	2	5	9	4	1
2	7	4	9	5	6	8	1	3
6	8	3	2	1	4	7	9	5
9	1	5	8	3	7	6	2	4

Model

- A variable for each cell, ranging from 1 to 9
- A 9x9 matrix of variables describing the problem
- Preassigned integers for the given hints
- `alldifferent` constraints for each row, column and 3x3 block

Reminder: `alldifferent`

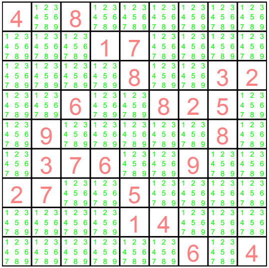
- Argument: list of variables
- Meaning: variables are pairwise different
- Reasoning: Forward Checking (FC)
  - When variable is assigned to value, remove the value from all other variables
  - If a variable has only one possible value, then it is assigned
  - If a variable has no possible values, then the constraint fails
  - Constraint is checked whenever one of its variables is assigned
  - Equivalent to decomposition into binary disequality constraints

Main Program

```
model(Matrix):-
    Matrix[1..9,1..9] :: 1..9,
    (for(I,1,9),
     param(Matrix) do
        alldifferent(Matrix[I,1..9]),
        alldifferent(Matrix[1..9,I])
    ),
    (multifor([I,J],[1,1],[7,7],[3,3]),
     param(Matrix) do
        alldifferent(flatten(Matrix[I..I+2,J..J+2]))
    ),
    flatten_array(Matrix,List),
    labeling(List).
```

Domain Visualizer

- Problem shown as matrix
- Each cell corresponds to a variable
- Instantiated: Shows integer value (large)
- Uninstantiated: Shows values in domain



Outline

Initial State (Forward Checking)

4	1 2 3 4 5 6 7 8 9	8	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1	7	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	8	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	3	2
1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	6	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	8	2	5	1 2 3 4 5 6 7 8 9
1 2 3 4 5 6 7 8 9	9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	8	1 2 3 4 5 6 7 8 9
1 2 3 4 5 6 7 8 9	3	7	6	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
2	7	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	5	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1	4	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	6	1 2 3 4 5 6 7 8 9	4

Propagation Steps (Forward Checking)

4	1 2 3 4 5 6 7 8 9	8	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1	7	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	8	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	3	2
1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	6	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	8	2	5	1 2 3 4 5 6 7 8 9
1 2 3 4 5 6 7 8 9	9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	8	1 2 3 4 5 6 7 8 9
1 2 3 4 5 6 7 8 9	3	7	6	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
2	7	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	5	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1	4	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	6	1 2 3 4 5 6 7 8 9	4

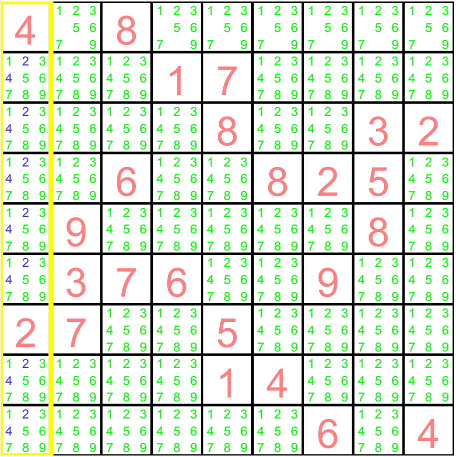
Skip Animation

Propagation Steps (Forward Checking)

4	1 2 3 4 5 6 7 8 9	8	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1	7	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	8	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	3	2
1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	6	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	8	2	5	1 2 3 4 5 6 7 8 9
1 2 3 4 5 6 7 8 9	9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	8	1 2 3 4 5 6 7 8 9
1 2 3 4 5 6 7 8 9	3	7	6	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
2	7	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	5	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1	4	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	6	1 2 3 4 5 6 7 8 9	4

◂ Back to Start ▸▸ Skip Animation

Propagation Steps (Forward Checking)



Back to Start   Skip Animation

Propagation Steps (Forward Checking)

4	1 2 3 5 6 7 9	8	1 2 3 5 6 7 9	1 2 3 5 6 7 9	1 2 3 5 6 7 9	1 2 3 5 6 7 9	1 2 3 5 6 7 9	1 2 3 5 6 7 9
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1 3 5 6 7 8 9	1 2 3 4 5 6 7 8 9	6	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	8	2	5	1 2 3 4 5 6 7 8 9
1 3 5 6 7 8 9	9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	8	1 2 3 4 5 6 7 8 9
1 3 5 6 7 8 9	3	7	6	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
2	7	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	5	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
1 3 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1	4	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
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◂ Back to Start   ▸▸ Skip Animation

Propagation Steps (Forward Checking)

4	1 2 3 5 6 7 9	8	1 2 3 5 6 7 9	1 2 3 5 6 7 9	1 2 3 5 6 7 9	1 2 3 5 6 7 9	1 2 3 5 6 7 9	1 2 3 5 6 7 9
3 5 6 8 9	2 3 4 5 6 8 9	2 3 4 5 6 8 9	1	7	2 3 4 5 6 8 9	2 3 4 5 6 8 9	2 3 4 5 6 8 9	2 3 4 5 6 8 9
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1 3 5 6 7 8 9	1 2 3 4 5 6 7 8 9	6	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	8	2	5	1 2 3 4 5 6 7 8 9
1 3 5 6 7 8 9	9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	8	1 2 3 4 5 6 7 8 9
1 3 5 6 7 8 9	3	7	6	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
2	7	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	5	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
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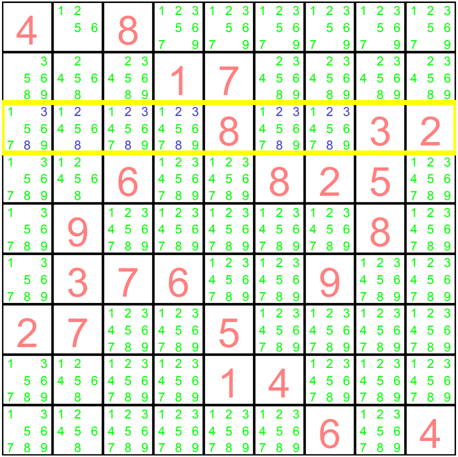
1 3 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	6	1 2 3 4 5 6 7 8 9	4

Back to Start
Skip Animation

Propagation Steps (Forward Checking)



Back to Start
Skip Animation

Propagation Steps (Forward Checking)

4	1 2 5 6	8	1 2 3 5 6 7 9	1 2 3 5 6 7 9	1 2 3 5 6 7 9	1 2 3 5 6 7 9	1 2 3 5 6 7 9	1 2 3 5 6 7 9
3 5 6 8 9	2 4 5 6 8	2 3 4 5 6 8 9	1	7	2 3 4 5 6 8 9	2 3 4 5 6 8 9	2 3 4 5 6 8 9	2 3 4 5 6 8 9
1 5 6 7 9	1 4 5 6	1 4 5 6 7 9	1 4 5 6 7 9	8	1 4 5 6 7 9	1 4 5 6 7 9	3	2
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1 3 5 6 7 8 9	9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	8	1 2 3 4 5 6 7 8 9
1 3 5 6 7 8 9	3	7	6	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
2	7	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	5	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
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1 3 5 6 7 8 9	1 2 4 5 6 8	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	6	1 2 3 4 5 6 7 8 9	4

Back to Start
Skip Animation

Propagation Steps (Forward Checking)

4	1 2 5 6	**8**	1 2 3 5 6 7 9	1 2 3 5 6 7 9	1 2 3 5 6 7 9	1 2 3 5 6 7 9	1 2 3 5 6 7 9	1 2 3 5 6 7 9
3 5 6 8 9	2 4 5 6 8	2 3 4 5 9	**1**	**7**	2 3 4 5 6 8 9	2 3 4 5 6 8 9	2 3 4 5 6 8 9	2 3 4 5 6 8 9
1 5 6 7 9	1 4 5 6	1 4 5 9	1 4 5 6 7 9	**8**	1 4 5 6 7 9	1 4 5 6 7 9	**3**	**2**
1 3 5 6 7 8 9	1 2 4 5 6 8	**6**	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	**8**	**2**	**5**	1 2 3 4 5 6 7 8 9
1 3 5 6 7 8 9	**9**	1 2 3 4 5 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	**8**	1 2 3 4 5 6 7 8 9
1 3 5 6 7 8 9	**3**	**7**	**6**	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	**9**	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
2	**7**	1 2 3 4 5 9	1 2 3 4 5 6 7 8 9	**5**	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
1 3 5 6 7 8 9	1 2 4 5 6 8	1 2 3 4 5 9	1 2 3 4 5 6 7 8 9	**1**	**4**	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
1 3 5 6 7 8 9	1 2 4 5 6 8	1 2 3 4 5 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	**6**	1 2 3 4 5 6 7 8 9	**4**

Back to Start | Skip Animation

Propagation Steps (Forward Checking)

4	1 2 5 6	8	1 2 3 5 6 7 9	1 2 3 5 6 7 9	1 2 3 5 6 7 9	1 2 3 5 6 7 9	1 2 3 5 6 7 9	1 2 3 5 6 7 9
3 5 6 8 9	2 4 5 6 8	2 3 4 5 9	1	7	2 3 4 5 6 8 9	2 3 4 5 6 8 9	2 3 4 5 6 8 9	2 3 4 5 6 8 9
1 5 6 7 9	1 4 5 6	1 4 5 9	1 4 5 6 7 9	8	1 4 5 6 7 9	1 4 5 6 7 9	3	2
1 3 7 9	1 4	6	1 3 4 7 9	1 3 4 7 9	8	2	5	1 3 4 7 9
1 3 5 6 7 8 9	9	1 2 3 4 5 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	8	1 2 3 4 5 6 7 8 9
1 3 5 6 7 8 9	3	7	6	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
2	7	1 2 3 4 5 9	1 2 3 4 5 6 7 8 9	5	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
1 3 5 6 7 8 9	1 2 4 5 6 8	1 2 3 4 5 9	1 2 3 4 5 6 7 8 9	1	4	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
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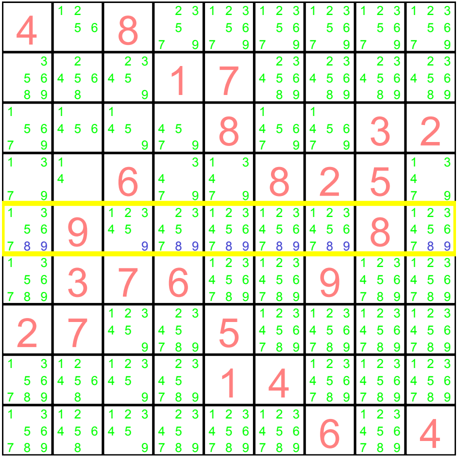
◂ Back to Start ▸▸ Skip Animation

Propagation Steps (Forward Checking)



Back to Start   Skip Animation

Propagation Steps (Forward Checking)

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1 5 6 7 9	1 4 5 6	1 4 5 9	4 5 7 9	8	1 4 5 6 7 9	1 4 5 6 7 9	3	2
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1 3 5 6 7	9	1 2 3 4 5	2 3 4 5 7	1 2 3 4 5 6 7	1 2 3 4 5 6 7	1 2 3 4 5 6 7	8	1 2 3 4 5 6 7
1 3 5 6 7 8 9	3	7	6	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
2	7	1 2 3 4 5 9	2 3 4 5 7 8 9	5	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
1 3 5 6 7 8 9	1 2 4 5 6 8	1 2 3 4 5 9	2 3 4 5 7 8 9	1	4	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
1 3 5 6 7 8 9	1 2 4 5 6 8	1 2 3 4 5 9	2 3 4 5 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	6	1 2 3 4 5 6 7 8 9	4

◂ Back to Start ▸▸ Skip Animation

Cork Constraint Computation Centre

Propagation Steps (Forward Checking)

4	1 2 5 6	8	2 3 5 7 9	2 3 6 9	1 2 3 5 6 7 9	1 2 3 5 6 7 9	1 2 3 5 6 7 9	1 2 3 5 6 7 9
3 5 6 8 9	2 4 5 6 8	2 3 4 5 9	1	7	2 3 4 5 6 8 9	2 3 4 5 6 8 9	2 3 4 5 6 8 9	2 3 4 5 6 8 9
1 5 6 7 9	1 4 5 6	1 4 5 9	4 5 7 9	8	1 4 5 6 7 9	1 4 5 6 7 9	3	2
1 3 7 9	1 4	6	3 4 7 9	3 4 9	8	2	5	1 3 4 7 9
1 3 5 6 7	9	1 2 3 4 5	2 3 4 5 7	2 3 4 6	1 2 3 4 5 6 7	1 2 3 4 5 6 7	8	1 2 3 4 5 6 7
1 3 5 6 7 8 9	3	7	6	2 3 4 6 9	1 2 3 4 5 6 7 8 9	9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
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◂ Back to Start ▸▸ Skip Animation

Propagation Steps (Forward Checking)

4	1 2 5 6	8	2 3 5 7 9	2 3 6 9	1 2 3 5 6 7 9	1 2 3 5 6 7 9	1 2 3 5 6 7 9	1 2 3 5 6 7 9
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◂ Back to Start ▸▸ Skip Animation

Propagation Steps (Forward Checking)

4	1 2 5 6	8	2 3 5 7 9	2 3 6 9	1 2 3 5 6 7 9	1 2 3 5 6 7 9	1 2 3 5 6 7 9	1 2 3 5 6 7 9
3 5 6 8 9	2 4 5 6 8	2 3 4 5 9	1	7	2 3 5 6 9	2 3 4 5 6 8 9	2 3 4 5 6 8 9	2 3 4 5 6 8 9
1 5 6 7 9	1 4 5 6	1 4 5 9	4 5 7 9	8	1 5 6 7 9	1 4 5 6 7 9	3	2
1 3 7 9	1 4	6	3 4 7 9	3 4 9	8	2	5	1 3 4 7 9
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1 5 8	3	7	6	2 4	1 2 5	9	1 2 4 5 8	1 2 4 5 8
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◂ Back to Start   ▸▸ Skip Animation

Propagation Steps (Forward Checking)

4	1 2 5 6	8	2 3 5 7 9	2 3 6 9	1 2 3 5 6 7 9	1 2 3 5 6 7 9	1 2 3 5 6 7 9	1 2 3 5 6 7 9
3 5 6 8 9	2 4 5 6 8	2 3 4 5 9	1	7	2 3 5 6 9	2 3 4 5 6 8 9	2 3 4 5 6 8 9	2 3 4 5 6 8 9
1 5 6 7 9	1 4 5 6	1 4 5 9	4 5 7 9	8	1 5 6 7 9	1 4 5 6 7 9	3	2
1 3 7 9	1 4	6	3 4 7 9	3 4 9	8	2	5	1 3 4 7 9
1 3 5 6 7	9	1 2 3 4 5	2 3 4 5 7	2 3 4 6	1 2 3 5 6 7	1 2 3 4 5 6 7	8	1 2 3 4 5 6 7
1 5 8	3	7	6	2 4	1 2 5	9	1 2 4 5 8	1 2 4 5 8
2	7	1 3 4 9	3 4 8 9	5	1 3 6 9	1 3 4 6 8 9	1 3 4 6 8 9	1 3 4 6 8 9
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1 3 5 6 7 8 9	1 2 4 5 6 8	1 2 3 4 5 9	2 3 4 5 7 8 9	2 3 4 6 9	1 2 3 5 6 7 9	6	1 2 3 4 5 6 7 8 9	4

Back to Start
Skip Animation

Propagation Steps (Forward Checking)

4	1 2 5 6	**8**	2 3 5 7 9	2 3 6 9	1 2 3 5 6 7 9	1 3 5 7	1 2 3 5 6 7 9	1 2 3 5 6 7 9
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1 3 5 6 7 8 9	1 2 4 5 6 8	1 2 3 4 5 9	2 3 4 5 7 8 9	2 3 4 6 9	1 2 3 5 6 7 9	**6**	1 2 3 4 5 6 7 8 9	**4**

◂ Back to Start   ▸▸ Skip Animation

Propagation Steps (Forward Checking)

4	1 2 5 6	8	2 3 5 7 9	2 3 6 9	1 2 3 5 6 7 9	1 3 5 7	1 2 3 5 6 7 9	1 2 3 5 6 7 9
3 5 6 8 9	2 4 5 6 8	2 3 4 5 9	1	7	2 3 5 6 9	3 4 5 8	2 3 4 5 6 8 9	2 3 4 5 6 8 9
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1 3 5 6 7 8 9	1 2 4 5 6 8	1 2 3 4 5 9	2 3 4 5 7 8 9	2 3 4 6 9	1 2 3 5 6 7 9	6	1 2 3 4 5 6 7 8 9	4

Back to Start
Skip Animation

Propagation Steps (Forward Checking)

4	1 2 5 6	8	2 3 5 7 9	2 3 6 9	1 2 3 5 6 7 9	1 3 5 7	1 2 6 7 9	1 2 3 5 6 7 9
3 5 6 8 9	2 4 5 6 8	2 3 4 5 9	1	7	2 3 5 6 9	3 4 5 8	2 4 6 9	2 3 4 5 6 8 9
1 5 6 7 9	1 4 5 6	1 4 5 9	4 5 7 9	8	1 5 6 7 9	1 4 5 7	3	2
1 3 7 9	1 4	6	3 4 7 9	3 4 9	8	2	5	1 3 4 7 9
1 3 5 6 7	9	1 2 3 4 5	2 3 4 5 7	2 3 4 6	1 2 3 5 6 7	1 3 4 5 7	8	1 2 3 4 5 6 7
1 5 8	3	7	6	2 4	1 2 5	9	1 2 4	1 2 4 5 8
2	7	1 3 4 9	3 4 8 9	5	1 3 6 9	1 3 4 8	1 4 6 9	1 3 4 6 8 9
3 5 6 7 8 9	2 5 6 8	2 3 5 9	2 3 5 7 8 9	1	4	3 5 7 8	2 6 7 9	2 3 5 6 7 8 9
1 3 5 6 7 8 9	1 2 4 5 6 8	1 2 3 4 5 9	2 3 4 5 7 8 9	2 3 4 6 9	1 2 3 5 6 7 9	6	1 2 4 6 7 9	4

Back to Start   Skip Animation

Propagation Steps (Forward Checking)

4	1 2 5 6	8	2 3 5 7 9	2 3 6 9	1 2 3 5 6 7 9	1 3 5 7	1 2 6 7 9	1 2 3 5 6 7 9
3 5 6 8 9	2 4 5 6 8	2 3 4 5 9	1	7	2 3 5 6 9	3 4 5 8	2 4 6 9	2 3 4 5 6 8 9
1 5 6 7 9	1 4 5 6	1 4 5 9	4 5 7 9	8	1 5 6 7 9	1 4 5 7	3	2
1 3 7 9	1 4	6	3 4 7 9	3 4 9	8	2	5	1 3 4 7 9
1 3 5 6 7	9	1 2 3 4 5	2 3 4 5 7	2 3 4 6	1 2 3 5 6 7	1 3 4 5 7	8	1 2 3 4 5 6 7
1 5 8	3	7	6	2 4	1 2 5	9	1 2 4	1 2 4 5 8
2	7	1 3 4 9	3 4 8 9	5	1 3 6 9	1 3 4 8	1 4 6 9	1 3 4 6 8 9
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◂ Back to Start   ▸▸ Skip Animation

Propagation Steps (Forward Checking)

4	1 2 5 6	8	2 3 5 7 9	2 3 6 9	1 2 3 5 6 7 9	1 3 5 7	1 2 6 7 9	1 3 5 6 7 9
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Back to Start   Skip Animation

Propagation Steps (Forward Checking)

4	1 2 5 6	8	2 3 5 7 9	2 3 6 9	1 2 3 5 6 7 9	1 3 5 7	1 2 6 7 9	1 3 5 6 7 9
3 5 6 9	2 5 6	2 3 5 9	1	7	2 3 5 6 9	3 4 5 8	2 4 6 9	3 5 6 8 9
1 5 6 7 9	1 5 6	1 5 9	4 5 7 9	8	1 5 6 7 9	1 4 5 7	3	2
1 3 7 9	1 4	6	3 4 7 9	3 4 9	8	2	5	1 3 7 9
1 3 5 6 7	9	1 2 3 4 5	2 3 4 5 7	2 3 4 6	1 2 3 5 6 7	1 3 4 5 7	8	1 3 5 6 7
1 5 8	3	7	6	2 4	1 2 5	9	1 2 4	1 5 8
2	7	1 3 4 9	3 4 8 9	5	1 3 6 9	1 3 4 8	1 4 6 9	1 3 6 8 9
3 5 6 7 8 9	2 5 6 8	2 3 5 9	2 3 5 7 8 9	1	4	3 5 7 8	2 6 7 9	3 5 6 7 8 9
1 3 5 7 8 9	1 2 5 8	1 2 3 5 9	2 3 5 7 8 9	2 3 9	1 2 3 5 7 9	6	1 2 7 9	4

Back to Start
Skip Animation

Propagation Steps (Forward Checking)

4	1 2 5 6	8	2 3 5 9	2 3 6 9	2 3 5 6 9	1 3 5 7	1 2 6 7 9	1 3 5 6 7 9
3 5 6 9	2 5 6	2 3 5 9	1	7	2 3 5 6 9	3 4 5 8	2 4 6 9	3 5 6 8 9
1 5 6 7 9	1 5 6	1 5 9	4 5 9	8	5 6 9	1 4 5 7	3	2
1 3 7 9	1 4	6	3 4 7 9	3 4 9	8	2	5	1 3 7 9
1 3 5 6 7	9	1 2 3 4 5	2 3 4 5 7	2 3 4 6	1 2 3 5 6 7	1 3 4 5 7	8	1 3 5 6 7
1 5 8	3	7	6	2 4	1 2 5	9	1 2 4	1 5 8
2	7	1 3 4 9	3 4 8 9	5	1 3 6 9	1 3 4 8	1 4 6 9	1 3 6 8 9
3 5 6 7 8 9	2 5 6 8	2 3 5 9	2 3 5 7 8 9	1	4	3 5 7 8	2 6 7 9	3 5 6 7 8 9
1 3 5 7 8 9	1 2 5 8	1 2 3 5 9	2 3 5 7 8 9	2 3 9	1 2 3 5 7 9	6	1 2 7 9	4

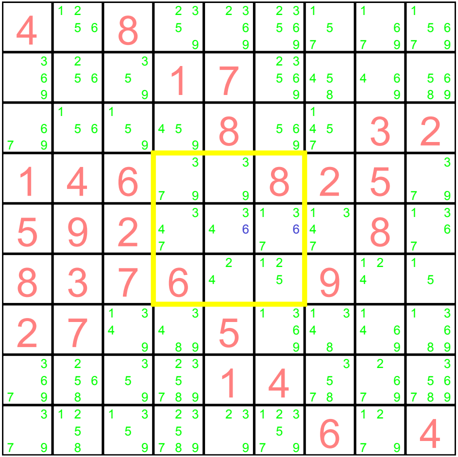
◂ Back to Start ▸▸ Skip Animation

Propagation Steps (Forward Checking)

4	{1 2 5 6}	8	{2 3 5 9}	{2 3 6 9}	{2 3 5 6 9}	{1 5 7}	{1 6 7 9}	{1 5 6 7 9}
{3 5 6 9}	{2 5 6}	{2 3 5 9}	1	7	{2 3 5 6 9}	{4 5 8}	{4 6 9}	{5 6 8 9}
{1 5 6 7 9}	{1 5 6}	{1 5 9}	{4 5 9}	8	{5 6 9}	{1 4 5 7}	3	2
1	4	6	{3 4 7 9}	{3 4 9}	8	2	5	{1 3 7 9}
5	9	2	{2 3 4 5 7}	{2 3 4 6}	{1 2 3 5 6 7}	{1 3 4 5 7}	8	{1 3 5 6 7}
8	3	7	6	{2 4}	{1 2 5}	9	{1 2 4}	{1 5 8}
2	7	{1 3 4 9}	{3 4 8 9}	5	{1 3 6 9}	{1 3 4 8}	{1 4 6 9}	{1 3 6 8 9}
{3 5 6 7 8 9}	{2 5 6 8}	{2 3 5 9}	{2 3 5 7 8 9}	1	4	{3 5 7 8}	{2 6 7 9}	{3 5 6 7 8 9}
{1 3 5 7 8 9}	{1 2 5 8}	{1 2 3 5 9}	{2 3 5 7 8 9}	{2 3 9}	{1 2 3 5 7 9}	6	{1 2 7 9}	4

◂ Back to Start   ▸▸ Skip Animation

Propagation Steps (Forward Checking)



◂ Back to Start ▸▸ Skip Animation

Propagation Steps (Forward Checking)

4	1 2 5 6	8	2 3 5 9	2 3 6 9	2 3 5 6 9	1 5 7	1 6 7 9	1 5 6 7 9
3 6 9	2 5 6	3 5 9	1	7	2 3 5 6 9	4 5 8	4 6 9	5 6 8 9
6 7 9	1 5 6	1 5 9	4 5 9	8	5 6 9	1 4 5 7	3	2
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3 6 7 9	2 5 6 8	3 5 9	2 3 5 7 8 9	1	4	3 5 7 8	2 6 7 9	3 5 6 7 8 9
3 7 9	1 2 5 8	1 3 5 9	2 3 5 7 8 9	2 3 9	1 2 3 5 7 9	6	1 2 7 9	4

Back to Start   Skip Animation

Propagation Steps (Forward Checking)

4	1 2 5 6	8	2 3 5 9	3 6 9	2 3 6 9	1 5 7	1 6 7 9	5 6 7 9
3 6 9	2 5 6	3 5 9	1	7	2 3 6 9	4 5 8	6 9	5 6 8 9
6 7 9	1 5 6	1 5 9	4 5 9	8	6 9	1 4 5 7	3	2
1	4	6	3 7 9	3 9	8	2	5	3 7
5	9	2	3 4 7	3 4	1 3 7	3 7	8	3 6 7
8	3	7	6	2	5	9	4	1
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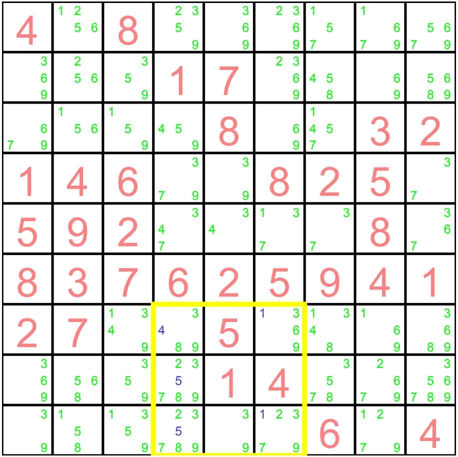
3 7 9	1 2 5 8	1 3 5 9	2 3 5 7 8 9	3 9	1 2 3 7 9	6	1 2 7 9	4

Back to Start Skip Animation

Propagation Steps (Forward Checking)



Back to Start   Skip Animation

Propagation Steps (Forward Checking)

4	1 2 5 6	8	2 3 5 9	3 6 9	2 3 6 9	1 5 7	1 6 7 9	5 6 7 9
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1	4	6	3 7 9	3 9	8	2	5	3 7
5	9	2	3 4 7	3 4	1 3 7	3 7	8	3 6 7
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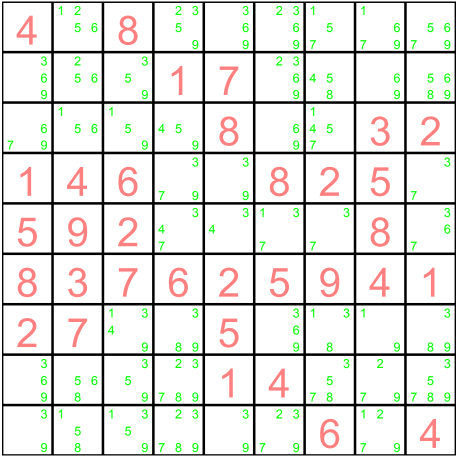
3 6 9	5 6 8	3 5 9	2 3 7 8 9	1	4	3 5 7 8	2 6 7 9	3 5 6 7 8 9
3 9	1 5 8	1 3 5 9	2 3 7 8 9	3 9	2 3 7 9	6	1 2 7 9	4

Back to Start
Skip Animation

Cork Constraint Computation Centre

Propagation Steps (Forward Checking)



Back to Start

After Setup (Forward Checking)

4	1 2 5 6	**8**	2 3 5 9	3 6 9	2 3 6 9	1 5 7	1 6 7 9	5 6 7 9
3 6 9	2 5 6	3 5 9	**1**	**7**	2 3 6 9	4 5 8	6 9	5 6 8 9
6 7 9	1 5 6	1 5 9	4 5 9	**8**	6 9	1 4 5 7	**3**	**2**
1	**4**	**6**	3 7 9	3 9	**8**	**2**	**5**	3 7
5	**9**	**2**	3 4 7	3 4	1 3 7	3 7	**8**	3 6 7
8	**3**	**7**	**6**	**2**	**5**	**9**	**4**	**1**
2	**7**	1 3 4 9	3 8 9	**5**	3 6 9	1 3 8	1 9	3 8 9
3 6 9	5 6 8	3 5 9	2 3 7 8 9	**1**	**4**	3 5 7 8	2 7 9	3 5 7 8 9
3 9	1 5 8	1 3 5 9	2 3 7 8 9	3 9	2 3 7 9	**6**	1 2 7 9	**4**

Outline

6 Problem

7 Initial Propagation (Forward Checking)

8 Improved Reasoning
- Domain Consistency
- Comparison

9 Search

Can we do better?

- The alldifferent constraint is missing propagation
  - How can we do more propagation?
  - Do we know when we derive all possible information from the constraint?
- Constraints only interact by changing domains of variables

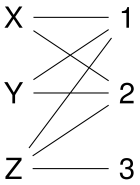
A Simpler Example

```
:-lib(ic).

top:-
    X :: 1..2,
    Y :: 1..2,
    Z :: 1..3,
    alldifferent([X,Y,Z]),
    writeln([X,Y,Z]).
```

Using Forward Checking

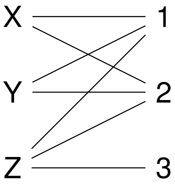
- No variable is assigned
- No reduction of domains
- But, values 1 and 2 can be removed from Z
- This means that Z is assigned to 3

Visualization of alldifferent as Graph



- Show problem as graph with two types of nodes
    - Variables on the left
    - Values on the right
- If value is in domain of variable, show link between them
- This is called a *bipartite* graph

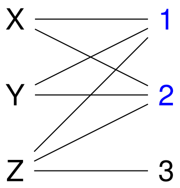
A Simpler Example



Value Graph for

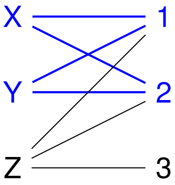
```
X :: 1..2,

Y :: 1..2,

Z :: 1..3
```

A Simpler Example



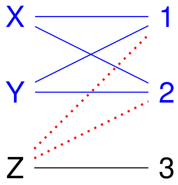
Check interval [1,2]

A Simpler Example



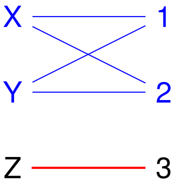
- Find variables completely contained in interval
- There are two: X and Y
- This uses up the capacity of the interval

A Simpler Example



No other variable can use that interval

A Simpler Example



Only one value left in domain of Z, this can be assigned

Idea (Hall Intervals)

- Take each interval of possible values, say size N
- Find all K variables whose domain is completely contained in interval
- If $K > N$ then the constraint is infeasible
- If $K = N$ then no other variable can use that interval
- Remove values from such variables if their bounds change
- If $K < N$ do nothing
- Re-check whenever domain bounds change

Implementation

- Problem: Too many intervals ($O(n^2)$) to consider
- Solution:
  - Check only those intervals which update bounds
  - Enumerate intervals incrementally
  - Starting from lowest(highest) value
  - Using sorted list of variables
- Complexity: $O(n \log(n))$ in standard implementations
- Important: Only looks at min/max bounds of variables

Bounds Consistency

Definition

A constraint achieves *bounds consistency*, if for the lower and upper bound of every variable, it is possible to find values for all other variables between their lower and upper bounds which satisfy the constraint.

Cork Constraint Computation Centre

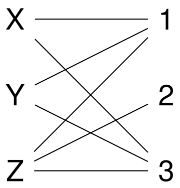
Can we do better?

- Bounds consistency only considers min/max bounds
- Ignores "holes" in domain
- Sometimes we can improve propagation looking at those holes

Another Simple Example

```
:-lib(ic).

top:-
     X :: [1,3],
     Y :: [1,3],
     Z :: 1..3,
     alldifferent([X,Y,Z]),
     writeln([X,Y,Z]).
```

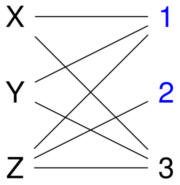
Another Simple Example



Value Graph for

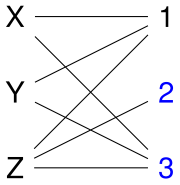
```
X :: [1,3],

Y :: [1,3],

Z :: 1..3
```

Another Simple Example



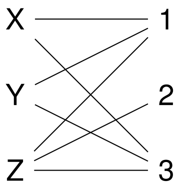
- Check interval [1,2]
- No domain of a variable completely contained in interval
- No propagation

Another Simple Example



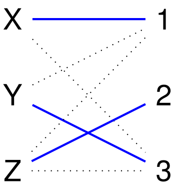
- Check interval [2,3]
- No domain of a variable completely contained in interval
- No propagation

Another Simple Example



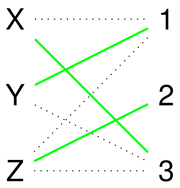
But, more propagation is possible, there are only two solutions

Another Simple Example



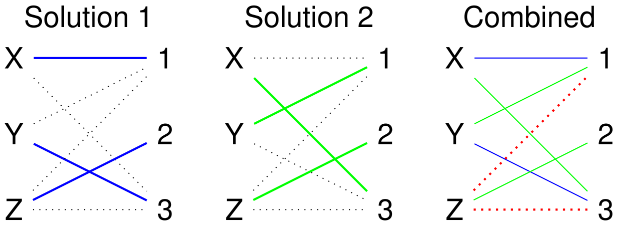
Solution 1: assignment in blue

Another Simple Example



Solution 2: assignment in green

Another Simple Example



Combining solutions shows that Z=1 and Z=3 are not possible.

Another Simple Example

Can we deduce this without enumerating solutions?

Solutions and maximal matchings

- A *Matching* is subset of edges which do not coincide in any node
- No matching can have more edges than number of variables
- Every solution corresponds to a *maximal matching* and vice versa
- If a link does not belong to some maximal matching, then it can be removed

Implementation

- Possible to compute all links which belong to some matching
  - Without enumerating all of them!
- Enough to compute **one** maximal matching
- Requires algorithm for *strongly connected components*
- Extra work required if more values than variables
- All links (values in domains) which are not supported can be removed
- Complexity: $O(n^{1.5}d)$

Domain Consistency

Definition

A constraint achieves *domain consistency*, if for every variable and for every value in its domain, it is possible to find values in the domains of all other variables which satisfy the constraint.

- Also called *generalized arc consistency (GAC)*
- or *hyper arc consistency*

Can we still do better?

- NO! This extracts all information from this one constraint
- We could perhaps improve speed, but not propagation
- But possible to use different model
- Or model interaction of multiple constraints

Should all constraints achieve domain consistency?

- Domain consistency is usually more expensive than bounds consistency
  - Overkill for simple problems
  - Nice to have choices
- For some constraints achieving domain consistency is NP-hard
  - We have to live with more restricted propagation

Initial State (Domain Consistency)

4	1 2 3 4 5 6 7 8 9	8	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1	7	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	8	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	3	2
1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	6	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	8	2	5	1 2 3 4 5 6 7 8 9
1 2 3 4 5 6 7 8 9	9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	8	1 2 3 4 5 6 7 8 9
1 2 3 4 5 6 7 8 9	3	7	6	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
2	7	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	5	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1	4	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	6	1 2 3 4 5 6 7 8 9	4

Propagation Steps (Domain Consistency)

4	1 2 3 4 5 6 7 8 9	8	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
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1 2 3 4 5 6 7 8 9	9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	8	1 2 3 4 5 6 7 8 9
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2	7	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	5	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
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1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	6	1 2 3 4 5 6 7 8 9	4

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Propagation Steps (Domain Consistency)

4	1 2 3 4 5 6 7 8 9	8	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
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1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	6	1 2 3 4 5 6 7 8 9	4

Back to Start   Skip Animation

Propagation Steps (Domain Consistency)

4	1 2 3 5 6 7 9	8	1 2 3 5 6 7 9	1 2 3 5 6 7 9	1 2 3 5 6 7 9	1 2 3 5 6 7 9	1 2 3 5 6 7 9	1 2 3 5 6 7 9
1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1	7	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	8	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	3	2
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1 2 3 4 5 6 7 8 9	9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	8	1 2 3 4 5 6 7 8 9
1 2 3 4 5 6 7 8 9	3	7	6	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
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Back to Start | Skip Animation

Cork Constraint Computation Centre

Propagation Steps (Domain Consistency)

4	1 2 3 5 6 7 9	8	1 2 3 5 6 7 9	1 2 3 5 6 7 9	1 2 3 5 6 7 9	1 2 3 5 6 7 9	1 2 3 5 6 7 9	1 2 3 5 6 7 9
1 3 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1	7	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
1 3 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	8	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	3	2
1 3 5 6 7 8 9	1 2 3 4 5 6 7 8 9	6	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	8	2	5	1 2 3 4 5 6 7 8 9
1 3 5 6 7 8 9	9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	8	1 2 3 4 5 6 7 8 9
1 3 5 6 7 8 9	3	7	6	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
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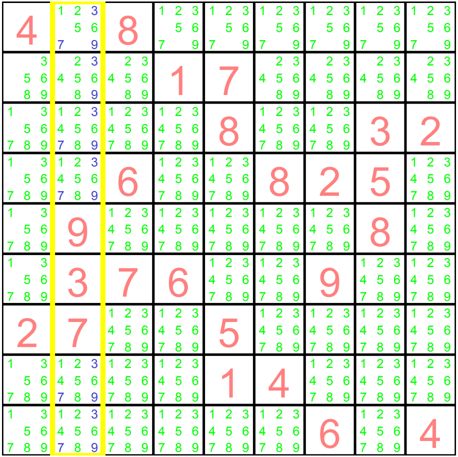
1 3 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1	4	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
1 3 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	6	1 2 3 4 5 6 7 8 9	4

◂ Back to Start
⏩ Skip Animation

Propagation Steps (Domain Consistency)



Back to Start   Skip Animation

Propagation Steps (Domain Consistency)

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3 5 6 8 9	2 4 5 6 8	2 3 4 5 6 8 9	1	7	2 3 4 5 6 8 9	2 3 4 5 6 8 9	2 3 4 5 6 8 9	2 3 4 5 6 8 9
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1 3 5 6 7 8 9	9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	8	1 2 3 4 5 6 7 8 9
1 3 5 6 7 8 9	3	7	6	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
2	7	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	5	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
1 3 5 6 7 8 9	1 2 4 5 6 8	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1	4	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
1 3 5 6 7 8 9	1 2 4 5 6 8	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	6	1 2 3 4 5 6 7 8 9	4

◂ Back to Start
▸▸ Skip Animation

Propagation Steps (Domain Consistency)

4	1 2 5 6	8	1 2 3 5 6 7 9	1 2 3 5 6 7 9	1 2 3 5 6 7 9	1 2 3 5 6 7 9	1 2 3 5 6 7 9	1 2 3 5 6 7 9
3 5 6 8 9	2 4 5 6 8	2 3 4 5 6 8 9	1	7	2 3 4 5 6 8 9	2 3 4 5 6 8 9	2 3 4 5 6 8 9	2 3 4 5 6 8 9
1 5 6 7 9	1 4 5 6	1 4 5 6 7 9	1 4 5 6 7 9	8	1 4 5 6 7 9	1 4 5 6 7 9	3	2
1 3 5 6 7 8 9	1 2 4 5 6 8	6	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	8	2	5	1 2 3 4 5 6 7 8 9
1 3 5 6 7 8 9	9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	8	1 2 3 4 5 6 7 8 9
1 3 5 6 7 8 9	3	7	6	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
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1 3 5 6 7 8 9	1 2 4 5 6 8	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1	4	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
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◂ Back to Start ▸▸ Skip Animation

Cork Constraint Computation Centre

Propagation Steps (Domain Consistency)

4	1 2 5 6	8	1 2 3 5 6 7 9	1 2 3 5 6 7 9	1 2 3 5 6 7 9	1 2 3 5 6 7 9	1 2 3 5 6 7 9	1 2 3 5 6 7 9
3 5 6 8 9	2 4 5 6 8	2 3 4 5 9	1	7	2 3 4 5 6 8 9	2 3 4 5 6 8 9	2 3 4 5 6 8 9	2 3 4 5 6 8 9
1 5 6 7 9	1 4 5 6	1 4 5 9	1 4 5 6 7 9	8	1 4 5 6 7 9	1 4 5 6 7 9	3	2
1 3 5 6 7 8 9	1 2 4 5 6 8	6	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	8	2	5	1 2 3 4 5 6 7 8 9
1 3 5 6 7 8 9	9	1 2 3 4 5 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	8	1 2 3 4 5 6 7 8 9
1 3 5 6 7 8 9	3	7	6	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
2	7	1 2 3 4 5 9	1 2 3 4 5 6 7 8 9	5	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
1 3 5 6 7 8 9	1 2 4 5 6 8	1 2 3 4 5 9	1 2 3 4 5 6 7 8 9	1	4	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
1 3 5 6 7 8 9	1 2 4 5 6 8	1 2 3 4 5 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	6	1 2 3 4 5 6 7 8 9	4

Back to Start
Skip Animation

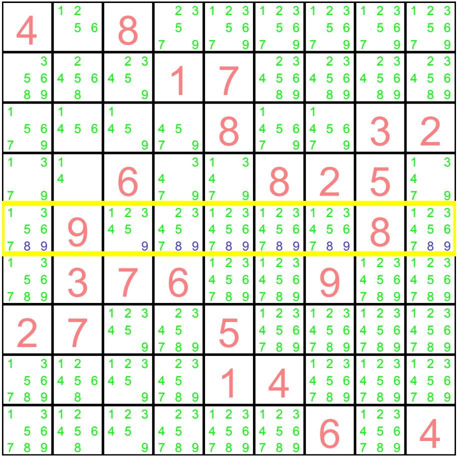
Cork Constraint Computation Centre

Propagation Steps (Domain Consistency)

4	1 2 5 6	8	1 2 3 5 6 7 9	1 2 3 5 6 7 9	1 2 3 5 6 7 9	1 2 3 5 6 7 9	1 2 3 5 6 7 9	1 2 3 5 6 7 9
3 5 6 8 9	2 4 5 6 8	2 3 4 5 9	1	7	2 3 4 5 6 8 9	2 3 4 5 6 8 9	2 3 4 5 6 8 9	2 3 4 5 6 8 9
1 5 6 7 9	1 4 5 6	1 4 5 9	1 4 5 6 7 9	8	1 4 5 6 7 9	1 4 5 6 7 9	3	2
1 3 7 9	1 4	6	1 3 4 7 9	1 3 4 7 9	8	2	5	1 3 4 7 9
1 3 5 6 7 8 9	9	1 2 3 4 5 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	8	1 2 3 4 5 6 7 8 9
1 3 5 6 7 8 9	3	7	6	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
2	7	1 2 3 4 5 9	1 2 3 4 5 6 7 8 9	5	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
1 3 5 6 7 8 9	1 2 4 5 6 8	1 2 3 4 5 9	1 2 3 4 5 6 7 8 9	1	4	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
1 3 5 6 7 8 9	1 2 4 5 6 8	1 2 3 4 5 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	6	1 2 3 4 5 6 7 8 9	4

◂ Back to Start   ▸▸ Skip Animation

Propagation Steps (Domain Consistency)



Back to Start
Skip Animation

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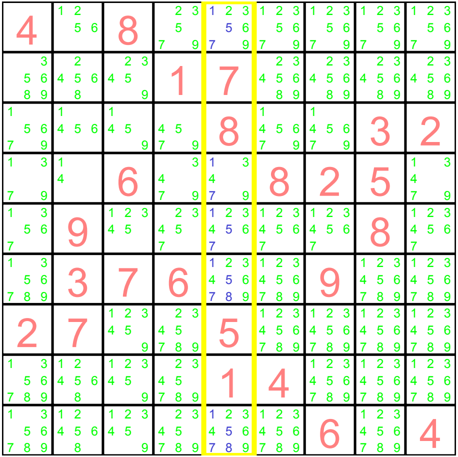

Back to Start
Skip Animation

Propagation Steps (Domain Consistency)

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3 5 6 8 9	2 4 5 6 8	2 3 4 5 9	1	7	2 3 4 5 6 8 9	2 3 4 5 6 8 9	2 3 4 5 6 8 9	2 3 4 5 6 8 9
1 5 6 7 9	1 4 5 6	1 4 5 9	4 5 7 9	8	1 4 5 6 7 9	1 4 5 6 7 9	3	2
1 3 7 9	1 4	6	3 4 7 9	3 4 9	8	2	5	1 3 4 7 9
1 3 5 6 7	9	1 2 3 4 5	2 3 4 5 7	2 3 4 6	1 2 3 4 5 6 7	1 2 3 4 5 6 7	8	1 2 3 4 5 6 7
1 3 5 6 7 8 9	3	7	6	2 3 4 6 9	1 2 3 4 5 6 7 8 9	9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
2	7	1 2 3 4 5 9	2 3 4 5 7 8 9	5	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
1 3 5 6 7 8 9	1 2 4 5 6 8	1 2 3 4 5 9	2 3 4 5 7 8 9	1	4	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
1 3 5 6 7 8 9	1 2 4 5 6 8	1 2 3 4 5 9	2 3 4 5 7 8 9	2 3 4 6 9	1 2 3 4 5 6 7 8 9	6	1 2 3 4 5 6 7 8 9	4

◂ Back to Start   ▸▸ Skip Animation

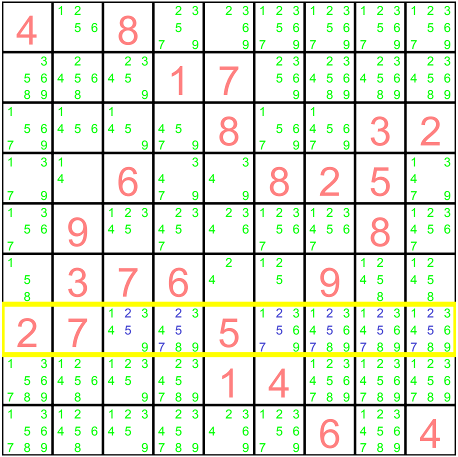
Cork Constraint Computation Centre

Propagation Steps (Domain Consistency)

4	1 2 5 6	8	2 3 5 7 9	2 3 6 9	1 2 3 5 6 7 9	1 2 3 5 6 7 9	1 2 3 5 6 7 9	1 2 3 5 6 7 9
3 5 6 8 9	2 4 5 6 8	2 3 4 5 9	1	7	2 3 4 5 6 8 9	2 3 4 5 6 8 9	2 3 4 5 6 8 9	2 3 4 5 6 8 9
1 5 6 7 9	1 4 5 6	1 4 5 9	4 5 7 9	8	1 4 5 6 7 9	1 4 5 6 7 9	3	2
1 3 7 9	1 4	6	3 4 7 9	3 4 9	8	2	5	1 3 4 7 9
1 3 5 6 7	9	1 2 3 4 5	2 3 4 5 7	2 3 4 6	1 2 3 4 5 6 7	1 2 3 4 5 6 7	8	1 2 3 4 5 6 7
1 5 8	3	7	6	2 4	1 2 4 5 8	9	1 2 4 5 8	1 2 4 5 8
2	7	1 2 3 4 5 9	2 3 4 5 7 8 9	5	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
1 3 5 6 7 8 9	1 2 4 5 6 8	1 2 3 4 5 9	2 3 4 5 7 8 9	1	4	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
1 3 5 6 7 8 9	1 2 4 5 6 8	1 2 3 4 5 9	2 3 4 5 7 8 9	2 3 4 6 9	1 2 3 4 5 6 7 8 9	6	1 2 3 4 5 6 7 8 9	4

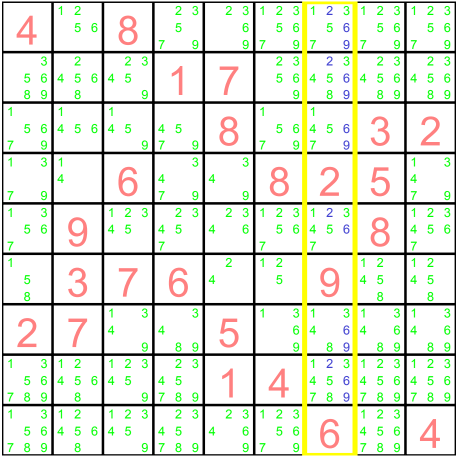
Back to Start   Skip Animation

Propagation Steps (Domain Consistency)



Back to Start | Skip Animation

Propagation Steps (Domain Consistency)



◂ Back to Start ▸▸ Skip Animation

Propagation Steps (Domain Consistency)

4	1 2 5 6	**8**	2 3 5 7 9	2 3 6 9	1 2 3 5 6 7 9	1 3 5 7	1 2 3 5 6 7 9	1 2 3 5 6 7 9
3 5 6 8 9	2 4 5 6 8	2 3 4 5 9	**1**	**7**	2 3 5 6 9	3 4 5 8	2 3 4 5 6 8 9	2 3 4 5 6 8 9
1 5 6 7 9	1 4 5 6	1 4 5 9	4 5 7 9	**8**	1 5 6 7 9	1 4 5 7	**3**	**2**
1 3 7 9	1 4	**6**	3 4 7 9	3 4 9	**8**	**2**	**5**	1 3 4 7 9
1 3 5 6 7	**9**	1 2 3 4 5	2 3 4 5 7	2 3 4 6	1 2 3 5 6 7	1 3 4 5 7	**8**	1 2 3 4 5 6 7
1 5 8	**3**	**7**	**6**	2 4	1 2 5	**9**	1 2 4 5 8	1 2 4 5 8
2	**7**	1 3 4 9	3 4 8 9	**5**	1 3 6 9	1 3 4 8	1 3 4 6 8 9	1 3 4 6 8 9
1 3 5 6 7 8 9	1 2 4 5 6 8	1 2 3 4 5 9	2 3 4 5 7 8 9	**1**	**4**	1 3 4 5 7 8	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
1 3 5 6 7 8 9	1 2 4 5 6 8	1 2 3 4 5 9	2 3 4 5 7 8 9	2 3 4 6 9	1 2 3 5 6 7 9	**6**	1 2 3 4 5 6 7 8 9	**4**

◂ Back to Start ▸▸ Skip Animation

Propagation Steps (Domain Consistency)

4	1 2 5 6	8	2 3 5 7 9	2 3 6 9	1 2 3 5 6 7 9	1 3 5 7	1 2 3 5 6 7 9	1 2 3 5 6 7 9
3 5 6 8 9	2 4 5 6 8	2 3 4 5 9	1	7	2 3 5 6 9	3 4 5 8	2 3 4 5 6 8 9	2 3 4 5 6 8 9
1 5 6 7 9	1 4 5 6	1 4 5 9	4 5 7 9	8	1 5 6 7 9	1 4 5 7	3	2
1 3 7 9	1 4	6	3 4 7 9	3 4 9	8	2	5	1 3 4 7 9
1 3 5 6 7	9	1 2 3 4 5	2 3 4 5 7	2 3 4 6	1 2 3 5 6 7	1 3 4 5 7	8	1 2 3 4 5 6 7
1 5 8	3	7	6	2 4	1 2 5	9	1 2 4 5 8	1 2 4 5 8
2	7	1 3 4 9	3 4 8 9	5	1 3 6 9	1 3 4 8	1 3 4 6 8 9	1 3 4 6 8 9
3 5 6 7 8 9	2 5 6 8	2 3 5 9	2 3 5 7 8 9	1	4	3 5 7 8	2 3 5 6 7 8 9	2 3 5 6 7 8 9
1 3 5 6 7 8 9	1 2 4 5 6 8	1 2 3 4 5 9	2 3 4 5 7 8 9	2 3 4 6 9	1 2 3 5 6 7 9	6	1 2 3 4 5 6 7 8 9	4

Back to Start
Skip Animation

Propagation Steps (Domain Consistency)

4	1 2 5 6	**8**	2 3 5 7 9	2 3 6 9	1 2 3 5 6 7 9	1 3 5 7	1 2 6 7 9	1 2 3 5 6 7 9
3 5 6 8 9	2 4 5 6 8	2 3 4 5 9	**1**	**7**	2 3 5 6 9	3 4 5 8	2 4 6 9	2 3 4 5 6 8 9
1 5 6 7 9	1 4 5 6	1 4 5 9	4 5 7 9	**8**	1 5 6 7 9	1 4 5 7	**3**	**2**
1 3 7 9	1 4	**6**	3 4 7 9	3 4 9	**8**	**2**	**5**	1 3 4 7 9
1 3 5 6 7	**9**	1 2 3 4 5	2 3 4 5 7	2 3 4 6	1 2 3 5 6 7	1 3 4 5 7	**8**	1 2 3 4 5 6 7
1 5 8	**3**	**7**	**6**	2 4	1 2 5	**9**	1 2 4	1 2 4 5 8
2	**7**	1 3 4 9	3 4 8 9	**5**	1 3 6 9	1 3 4 8	1 4 6 9	1 3 4 6 8 9
3 5 6 7 8 9	2 5 6 8	2 3 5 9	2 3 5 7 8 9	**1**	**4**	3 5 7 8	2 6 7 9	2 3 5 6 7 8 9
1 3 5 6 7 8 9	1 2 4 5 6 8	1 2 3 4 5 9	2 3 4 5 7 8 9	2 3 4 6 9	1 2 3 5 6 7 9	**6**	1 2 4 6 7 9	**4**

Back to Start
Skip Animation

Propagation Steps (Domain Consistency)

4	1 2 5 6	8	2 3 5 7 9	2 3 6 9	1 2 3 5 6 7 9	1 3 5 7	1 2 6 7 9	1 2 3 5 6 7 9
3 5 6 8 9	2 4 5 6 8	2 3 4 5 9	1	7	2 3 5 6 9	3 4 5 8	2 4 6 9	2 3 4 5 6 8 9
1 5 6 7 9	1 4 5 6	1 4 5 9	4 5 7 9	8	1 5 6 7 9	1 4 5 7	3	2
1 3 7 9	1 4	6	3 4 7 9	3 4 9	8	2	5	1 3 4 7 9
1 3 5 6 7	9	1 2 3 4 5	2 3 4 5 7	2 3 4 6	1 2 3 5 6 7	1 3 4 5 7	8	1 2 3 4 5 6 7
1 5 8	3	7	6	2 4	1 2 5	9	1 2 4	1 2 4 5 8
2	7	1 3 4 9	3 4 8 9	5	1 3 6 9	1 3 4 8	1 4 6 9	1 3 4 6 8 9
3 5 6 7 8 9	2 5 6 8	2 3 5 9	2 3 5 7 8 9	1	4	3 5 7 8	2 6 7 9	2 3 5 6 7 8 9
1 3 5 7 8 9	1 2 5 8	1 2 3 5 9	2 3 5 7 8 9	2 3 9	1 2 3 5 7 9	6	1 2 7 9	4

Back to Start
Skip Animation

Propagation Steps (Domain Consistency)

4	1 2 5 6	8	2 3 5 7 9	2 3 6 9	1 2 3 5 6 7 9	1 3 5 7	1 2 6 7 9	1 3 5 6 7 9
3 5 6 8 9	2 4 5 6 8	2 3 4 5 9	1	7	2 3 5 6 9	3 4 5 8	2 4 6 9	3 5 6 8 9
7	1 4 5 6	1 4 5 9	4 5 7 9	8	1 5 6 7 9	1 4 5 7	3	2
1 3 7 9	4	6	3 4 7 9	3 4 9	8	2	5	1 3 7 9
1 3 5 6 7	9	1 2 3 4 5	2 3 4 5 7	2 3 4 6	1 2 3 5 6 7	1 3 4 5 7	8	1 3 5 6 7
1 5 8	3	7	6	2 4	1 2 5	9	1 2 4	1 5 8
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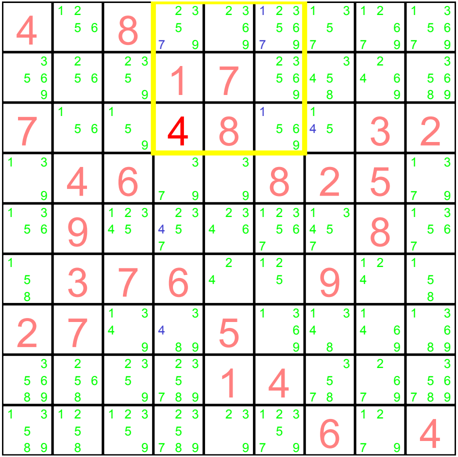
3 5 6 7 8 9	2 5 6 8	2 3 5 9	2 3 5 7 8 9	1	4	3 5 7 8	2 6 7 9	3 5 6 7 8 9
1 3 5 7 8 9	1 2 5 8	1 2 3 5 9	2 3 5 7 8 9	2 3 9	1 2 3 5 7 9	6	1 2 7 9	4

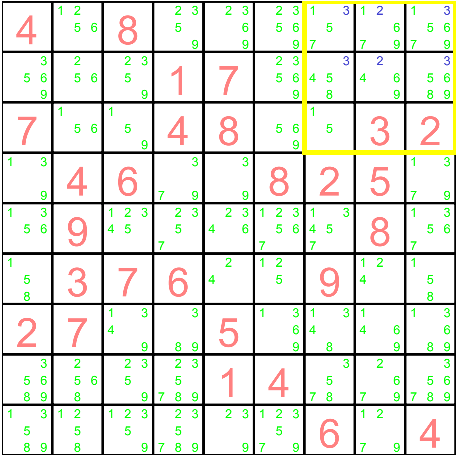
◂ Back to Start ▸▸ Skip Animation

Propagation Steps (Domain Consistency)



◂ Back to Start   ▸▸ Skip Animation

Propagation Steps (Domain Consistency)



Back to Start   Skip Animation

Propagation Steps (Domain Consistency)

4	1 2 5 6	8	2 3 5 9	2 3 6 9	2 3 5 6 9	1 5 7	1 6 7 9	1 5 6 7 9
3 5 6 9	2 5 6	2 3 5 9	1	7	2 3 5 6 9	4 5 8	4 6 9	5 6 8 9
7	1 5 6	1 5 9	4	8	5 6 9	1 5	3	2
1	4	6	3 7 9	3 9	8	2	5	1 3 7 9
5	9	2	2 3 5 7	2 3 4 6	1 2 3 5 6 7	1 3 4 5 7	8	1 3 5 6 7
8	3	7	6	2 4	1 2 5	9	1 2 4	1 5 8
2	7	4	3 8 9	5	1 3 6 9	1 3 4 8	1 4 6 9	1 3 6 8 9
3 5 6 8 9	2 5 6 8	2 3 5 9	2 3 5 7 8 9	1	4	3 5 7 8	2 6 7 9	3 5 6 7 8 9
1 3 5 8 9	1 2 5 8	1 2 3 5 9	2 3 5 7 8 9	2 3 9	1 2 3 5 7 9	6	1 2 7 9	4

Back to Start   Skip Animation

Propagation Steps (Domain Consistency)

4	2	8	5	6	3	1	1 6 7 9	1 5 6 7 9
3 6 9	5	3 5 9	1	7	2	4	6	8
7	6	1	4	8	9	5	3	2
1	4	6	3 7 9	3 9	8	2	5	3 7 9
5	9	2	3 7	4	1	1 3 4 7	8	6
8	3	7	6	2	5	9	4	1
2	7	4	3 8 9	5	6	8	1	1 3 6 8 9
6	8	3 5 9	2	1	4	3 5 7 8	2 6 7 9	5
3 9	1	5	8	2 3 9	7	6	2	4

Back to Start
Skip Animation

Propagation Steps (Domain Consistency)

4	2	8	5	6	3	1	{7, 9}	{7, 9}
{3, 9}	5	{3, 9}	1	7	2	4	6	8
7	6	1	4	8	9	5	3	2
1	4	6	{3, 7, 9}	{3, 9}	8	2	5	{3, 7, 9}
5	9	2	{3, 7}	4	1	{3, 7}	8	6
8	3	7	6	2	5	9	4	1
2	7	4	{3, 9}	5	6	8	1	{3, 9}
6	8	{3, 9}	2	1	4	{3, 7}	{7, 9}	5
{3, 9}	1	5	8	{3, 9}	7	6	2	4

◂ Back to Start ▸▸ Skip Animation

Propagation Steps (Domain Consistency)

4	2	8	5	6	3	1	7 9	7 9
3 9	5	3 9	1	7	2	4	6	8
7	6	1	4	8	9	5	3	2
1	4	6	3 7 9	3 9	8	2	5	3 7
5	9	2	3 7	4	1	3 7	8	6
8	3	7	6	2	5	9	4	1
2	7	4	3 9	5	6	8	1	3 9
6	8	3 9	2	1	4	3 7	7 9	5
3 9	1	5	8	3 9	7	6	2	4

Back to Start
Skip Animation

Propagation Steps (Domain Consistency)

4	2	8	5	6	3	1	7 9	7 9
3 9	5	3 9	1	7	2	4	6	8
7	6	1	4	8	9	5	3	2
1	4	6	3 7 9	3 9	8	2	5	3 7
5	9	2	3 7	4	1	3 7	8	6
8	3	7	6	2	5	9	4	1
2	7	4	3 9	5	6	8	1	3 9
6	8	3 9	2	1	4	3 7	7 9	5
3 9	1	5	8	3 9	7	6	2	4

Back to Start · Skip Animation

Cork Constraint Computation Centre

Propagation Steps (Domain Consistency)

4	2	8	5	6	3	1	7 9	7 9
3 9	5	3 9	1	7	2	4	6	8
7	6	1	4	8	9	5	3	2
1	4	6	3 7 9	3 9	8	2	5	3 7
5	9	2	3 7	4	1	3 7	8	6
8	3	7	6	2	5	9	4	1
2	7	4	3 9	5	6	8	1	3 9
6	8	3 9	2	1	4	3 7	7 9	5
3 9	1	5	8	3 9	7	6	2	4

Back to Start
Skip Animation

Propagation Steps (Domain Consistency)

4	2	8	5	6	3	1	7 9	7 9
3 9	5	3 9	1	7	2	4	6	8
7	6	1	4	8	9	5	3	2
1	4	6	3 7 9	3 9	8	2	5	3 7
5	9	2	3 7	4	1	3 7	8	6
8	3	7	6	2	5	9	4	1
2	7	4	3 9	5	6	8	1	3 9
6	8	3 9	2	1	4	3 7	7 9	5
3 9	1	5	8	3 9	7	6	2	4

Back to Start

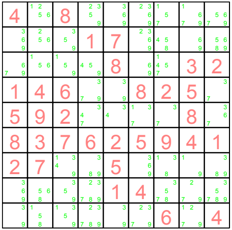
Cork Constraint Computation Centre

After Setup (Domain Consistency)

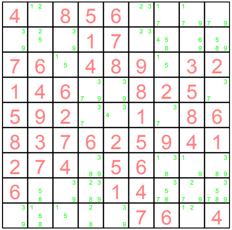
4	2	8	5	6	3	1	{7, 9}	{7, 9}
{3, 9}	5	{3, 9}	1	7	2	4	6	8
7	6	1	4	8	9	5	3	2
1	4	6	{3, 7, 9}	{3, 9}	8	2	5	{3, 7}
5	9	2	{3, 7}	4	1	{3, 7}	8	6
8	3	7	6	2	5	9	4	1
2	7	4	{3, 9}	5	6	8	1	{3, 9}
6	8	{3, 9}	2	1	4	{3, 7}	{7, 9}	5
{3, 9}	1	5	8	{3, 9}	7	6	2	4

Comparison

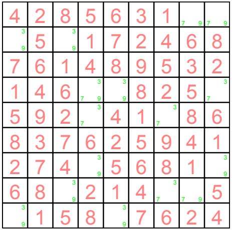
Forward Checking



Bounds Consistency



Domain Consistency



Typical?

- This does not always happen
- Sometimes, two methods produce same amount of propagation
- Possible to predict in certain special cases
- In general, tradeoff between speed and propagation
- Not always fastest to remove inconsistent values early
- But often required to find a solution at all

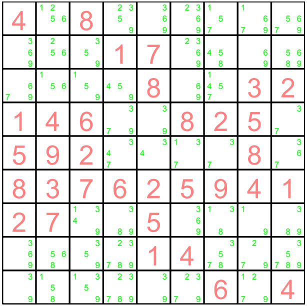
Outline

Simple search routine

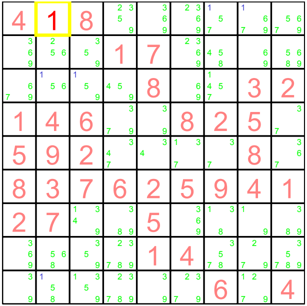
- Enumerate variables in given order
- Try values starting from smallest one in domain
- Complete, chronological backtracking

Search Tree (Forward Checking)

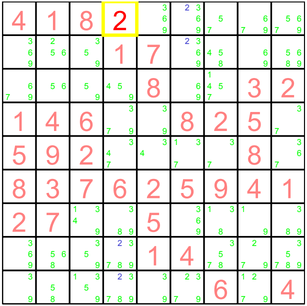
2



Search Tree (Forward Checking)



Search Tree (Forward Checking)



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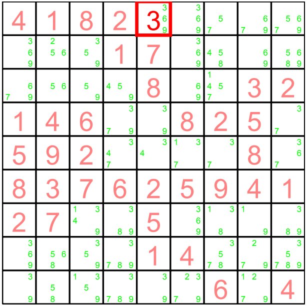

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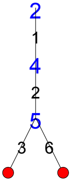

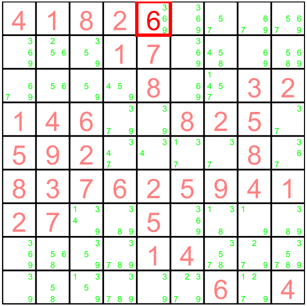

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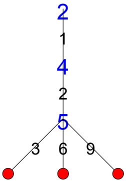

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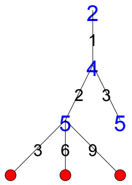
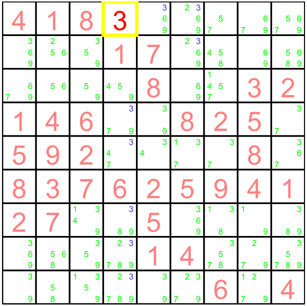
3 6 9	2 5 6	3 5 9	1	7	3 6 9	4 5 8	6 9	5 6 8 9
6 7 9	5 6	5 9	4 5 9	8	6 9	1 4 5 7	3	2
1	4	6	3 7 9	3 9	8	2	5	3 7
5	9	2	3 4 7	3 4	1 3 7	3 7	8	3 6 7
8	3	7	6	2	5	9	4	1
2	7	1 3 4 9	3 8 9	5	3 6 9	1 3 8	1 9	3 8 9
3 6 9	5 6 8	3 5 9	3 7 8 9	1	4	3 5 7 8	2 7 9	3 5 7 8 9
3 9	5 8	1 3 5 9	3 7 8 9	3 9	2 3 7 9	6	1 2 7 9	4

Search Tree (Forward Checking)





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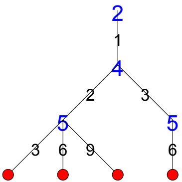

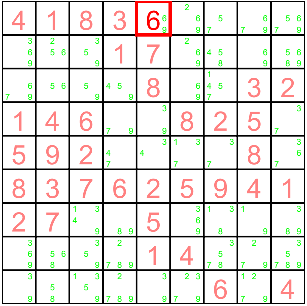

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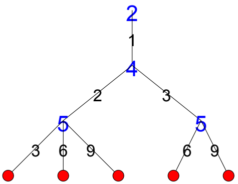

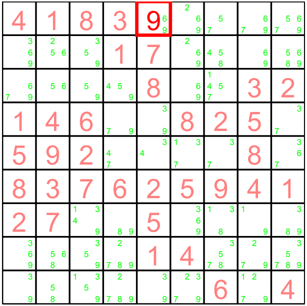

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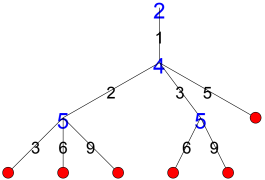

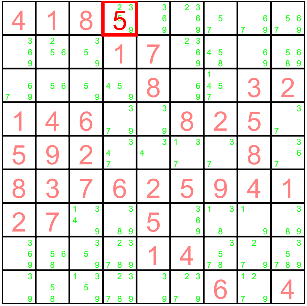

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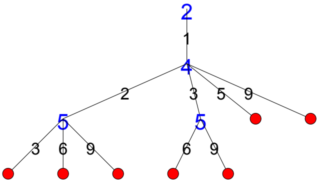

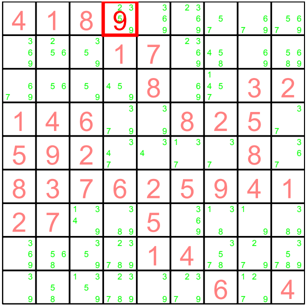

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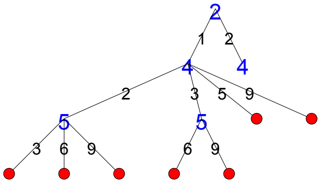

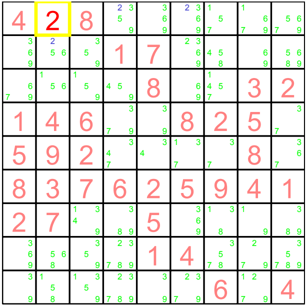

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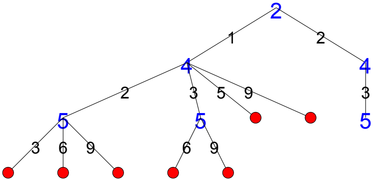

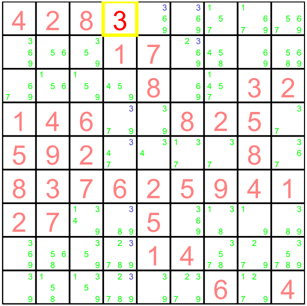

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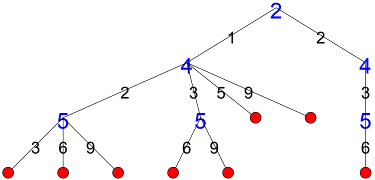

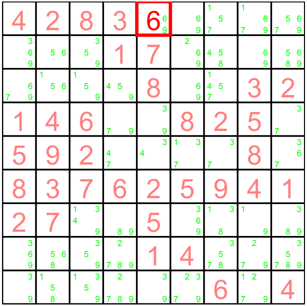

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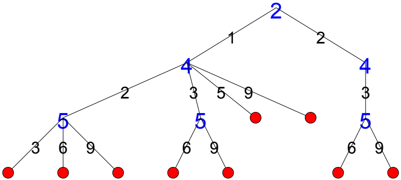
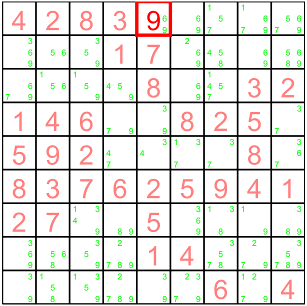
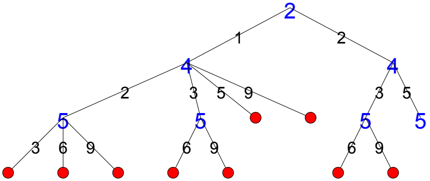
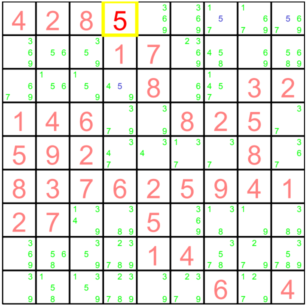




Cork Constraint Computation Centre

Search Tree (Forward Checking)





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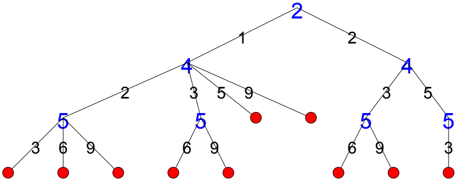

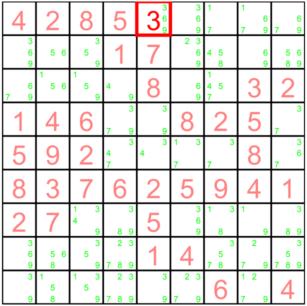

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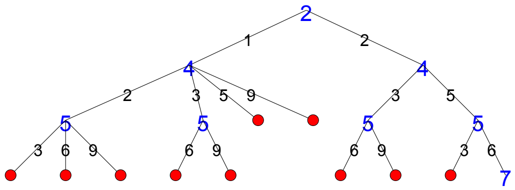

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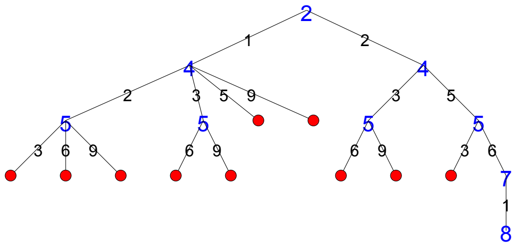
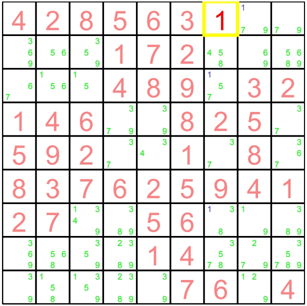
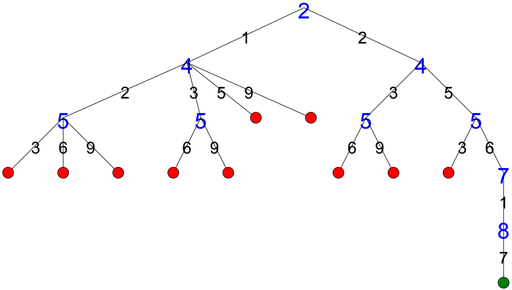




Cork Constraint Computation Centre

Search Tree (Forward Checking)



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3	5	9	1	7	2	4	6	8
7	6	1	4	8	9	5	3	2
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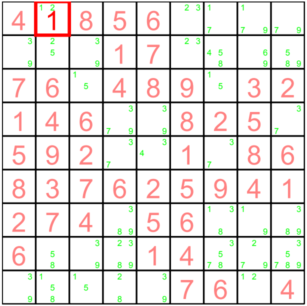
5	9	2	7	4	1	3	8	6
8	3	7	6	2	5	9	4	1
2	7	4	9	5	6	8	1	3
6	8	3	2	1	4	7	9	5
9	1	5	8	3	7	6	2	4

Search Tree (Bounds Consistency)

2

4	1 2	8	5	6	2 3	1 7	1 7 9	7 9
3 9	2 5	3 9	1	7	2 3	4 5 8	6 9	5 8 9
7	6	1 5	4	8	9	1 5	3	2
1	4	6	3 7 9	3 9	8	2	5	3 7
5	9	2	3 7	3 4	1	3 7	8	6
8	3	7	6	2	5	9	4	1
2	7	4	3 8 9	5	6	1 3 8	1 9	3 8 9
6	5 8	3 9	2 3 8 9	1	4	3 5 7 8	2 7 9	3 5 7 8 9
3 9	1 5 8	1 5	2 8	3 9	7	6	1 2	4

Search Tree (Bounds Consistency)



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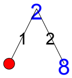

4	2	8	5	6	3	1	1 7 9	7 9
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7	6	1	4	8	9	5	3	2
1	4	6	3 7 9	3 9	8	2	5	3 7
5	9	2	3 7	3 4	1	3 7	8	6
8	3	7	6	2	5	9	4	1
2	7	4	3 8 9	5	6	8	1	3 8 9
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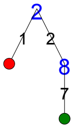

4	2	8	5	6	3	1	7	9
3	5	9	1	7	2	4	6	8
7	6	1	4	8	9	5	3	2
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5	9	2	7	4	1	3	8	6
8	3	7	6	2	5	9	4	1
2	7	4	9	5	6	8	1	3
6	8	3	2	1	4	7	9	5
9	1	5	8	3	7	6	2	4

Search Tree (Domain Consistency)

8

4	2	8	5	6	3	1	(7 9)	(7 9)
(3 9)	5	(3 9)	1	7	2	4	6	8
7	6	1	4	8	9	5	3	2
1	4	6	(3 7 9)	(3 9)	8	2	5	(3 7)
5	9	2	(3 7)	4	1	(3 7)	8	6
8	3	7	6	2	5	9	4	1
2	7	4	(3 9)	5	6	8	1	(3 9)
6	8	(3 9)	2	1	4	(3 7)	(7 9)	5
(3 9)	1	5	8	(3 9)	7	6	2	4

Search Tree (Domain Consistency)

4	2	8	5	6	3	1	7	9
3	5	9	1	7	2	4	6	8
7	6	1	4	8	9	5	3	2
1	4	6	3	9	8	2	5	7
5	9	2	7	4	1	3	8	6
8	3	7	6	2	5	9	4	1
2	7	4	9	5	6	8	1	3
6	8	3	2	1	4	7	9	5
9	1	5	8	3	7	6	2	4

Tradeoff: nodes and effort

- How many nodes do we need to explore?
- How much effort do we spend in each node?
- Extreme 1: SAT, do very little reasoning in each node, but do many nodes very rapidly
- Extreme 2: MIP, do a lot of reasoning in root node, and in each node, reduce number of nodes to explore
- Constraint Programming: Choice of balance tuneable for problem

Global Constraint Catalog

- `http://www.emn.fr/z-info/sdemasse/gccat/index.html`
- Description of 399 global constraints, 3250 pages
- Not all of them are widely used
- Detailed, meta-data description of constraints in Prolog

Key Global Constraints

- alldifferent
- cumulative
- cycle
- diffn
- element
- global_cardinality
- minimum_weight_alldifferent
- nvalue
- sort

`alldifferent(L)`

- A collection of variables *L* are pairwise different
- Algorithm: Flow, refinements
- Use: Everywhere
- Similar: `permutation`, `alldifferent_except_0`, `lex_alldifferent`

`cumulative(Tasks, Limit)`

- A set of tasks with start times s_i, durations d_i and resource requirements r_i do not exceed resource limit *Limit* at any time
- Algorithm: compulsory parts, energy, edge-finding, not-first/not-last
- Use: Scheduling, Placement
- Similar: `disjunctive`

`cycle(N, Succ)`

- A graph given by the successors s_i of nodes i contains N cycles
- Algorithm: `alldifferent`, strongly connected components
- Use: Transportation, Scheduling
- Similar: `circuit`, `tree`

`diffn(Obj)`

- n-dimensional objects given by orign $< x_{i1}, ..., x_{in} >$ and size $< d_{i1}, ..., d_{in} >$ do not overlap
- Algorithm: sweep, compulsory parts
- Use: placement
- Similar: `geost`

`element(X, L, C)`

- C is the x_{th} element of L
- Algorithm: basic
- Use: functional dependencies, cost
- Similar: `table`

global_cardinality(L, Values)

- Count how often certain values occur in the collection of variables *L*
- Algorithm: Flow, refinements
- Use: TImetabling
- Similar: generalizes `alldifferent`, `among_seq`

Cork Constraint Computation Centre

minimum_weight_alldifferent(L, Matrix, Cost)

- *Cost* is the cost of the assignment of the variables in *L*, the cost of each entry is given by the *Matrix* of cost values. The entries in *L* are pairwise different.
- Algorithm: Hungarian Method, Flow, Simplex
- Use: Resource allocation
- Similar: global_cardinality_with_costs

`nvalue(N, L)`

- Count the number *N* of distinct values in a collection of variables *L*
- Algorithm: specific, bounds-consistency only
- Use: Assignment problems
- Similar: `same`

`sort(L,K)`

- K is the sorted collection of the variables in L
- Algorithm: specific
- Use: Building block for reformulation of other constraints

Why are there so many global constraints?

- Algorithmic aspect
  - More specific restrictions allow more refined algorithms
- Modelling aspect
  - Capture exactly the properties of the problem we are after
- Families of constraints, restrictions and generalizations

Constraint Programming to the rescue!

- Constraint Seeker tool (Beldiceanu, Simonis 2011)
- `http://seeker.mines-nantes.fr/`
- Given positive and negative examples, produce ranked list of possible matching global constraints
- Itself a collection of constraint programs

Basis of modelling tool

- Model Seeker (Beldiceanu, Simonis 2012)
- From positive sample solutions find potential models for problem
- Expressed as conjunction of global constraints
- For highly structured problems
- Do **you** have sample solutions?

Part III

Customizing Search

What we want to introduce

- Importance of search strategy, constraints alone are not enough
- Dynamic variable ordering exploits information from propagation
- Variable and value choice
- Hard to find strategy which works all the time
- `search` builtin, flexible search abstraction
- Different way of improving stability of search routine

Example Problem

- N-Queens puzzle
- Rather weak constraint propagation
- Many solutions, limited number of symmetries
- Easy to scale problem size

Outline

Problem Definition

8-Queens

Place 8 queens on an 8×8 chessboard so that no queen attacks another. A queen attacks all cells in horizontal, vertical and diagonal direction. Generalizes to boards of size $N \times N$.

Problem Definition

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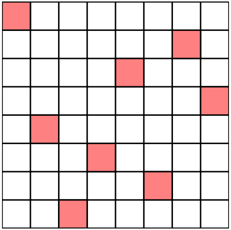


Solution for board size 8×8

Outline

10 Problem

11 Program
- Model

12 Naive Search

13 Improvements

Basic Model

- Cell based Model
  - A 0/1 variable for each cell to say if it is occupied or not
  - Constraints on rows, columns and diagonals to enforce no-attack
  - N^2 variables, $6N - 2$ constraints
- Column (Row) based Model
  - A 1..N variable for each column, stating position of queen in the column
  - Based on observation that each column must contain exactly one queen
  - N variables, $N^2/2$ binary constraints

Model

$$\text{assign} \quad [X_1, X_2, ...X_N]$$

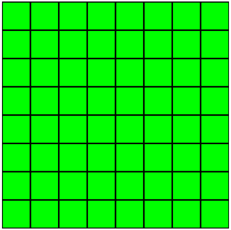
s.t.

$$\forall 1 \leq i \leq N: \quad X_i \in 1..N$$
$$\forall 1 \leq i < j \leq N: \quad X_i \neq X_j$$
$$\forall 1 \leq i < j \leq N: \quad X_i \neq X_j + i - j$$
$$\forall 1 \leq i < j \leq N: \quad X_i \neq X_j + j - i$$

Outline

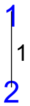
Default Strategy

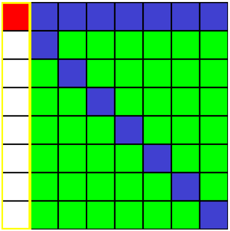
1



Skip Animation

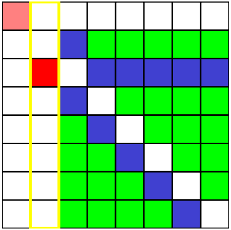
Default Strategy





Back to Start
Skip Animation

Default Strategy



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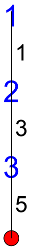

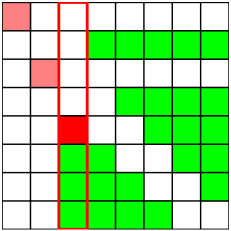

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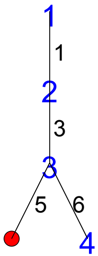

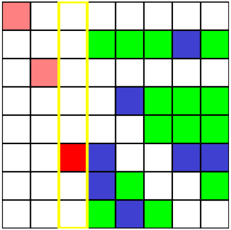

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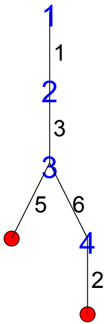

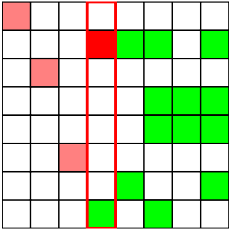

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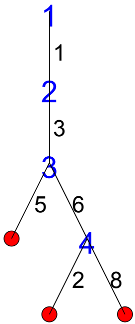

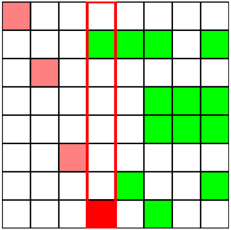

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Skip Animation

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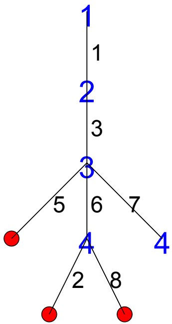

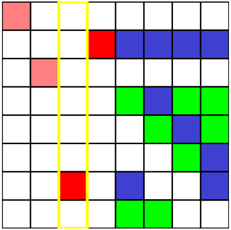

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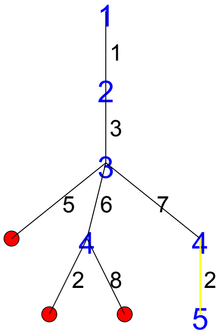

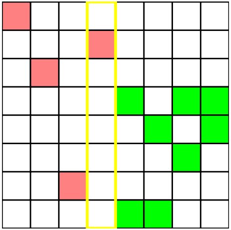

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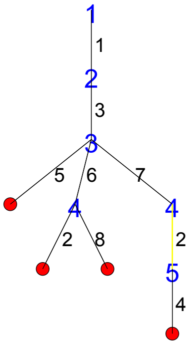

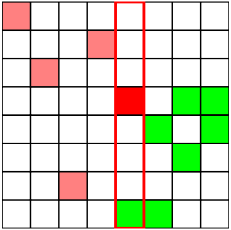

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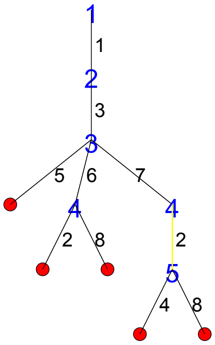

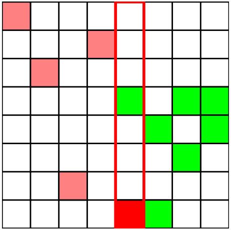

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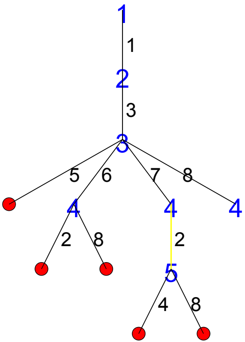

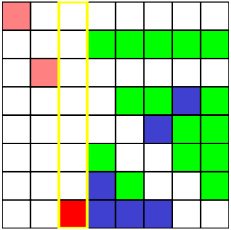

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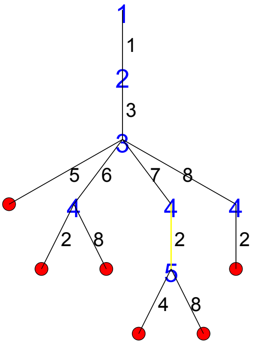

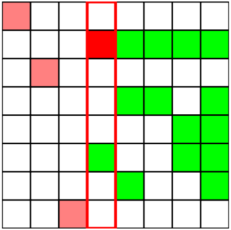

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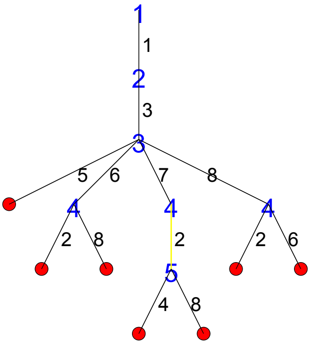

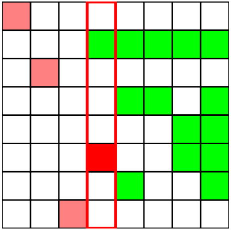

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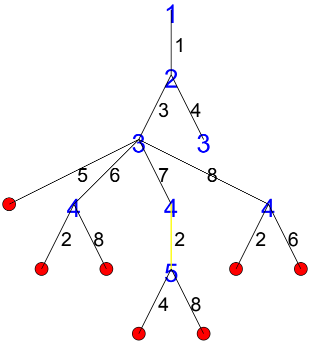

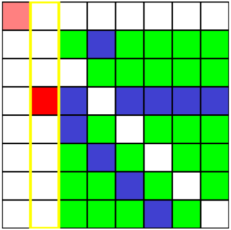

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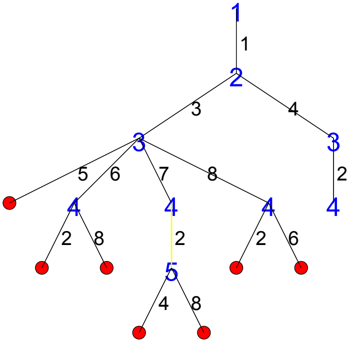

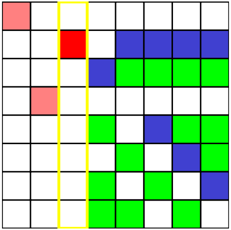

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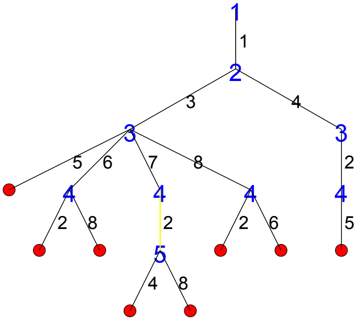

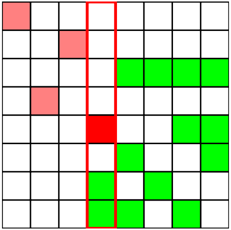

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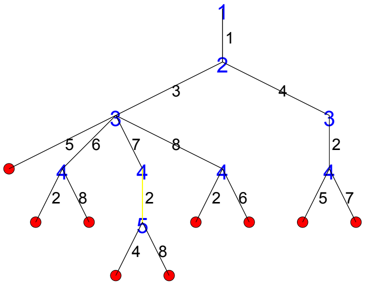

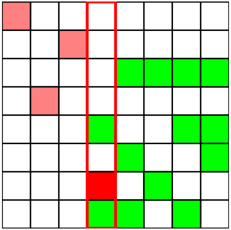

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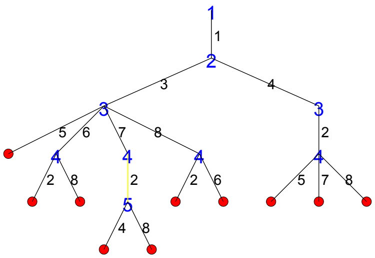

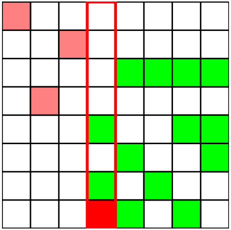

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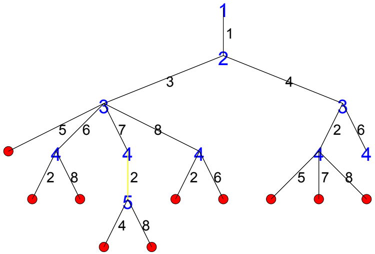

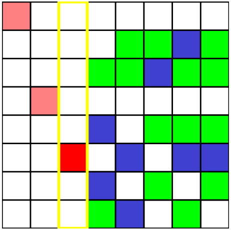

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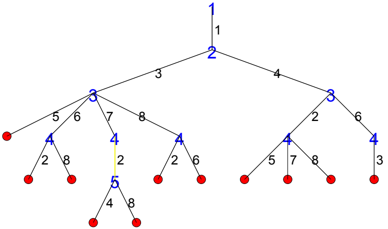
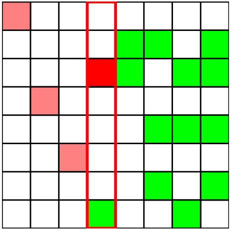
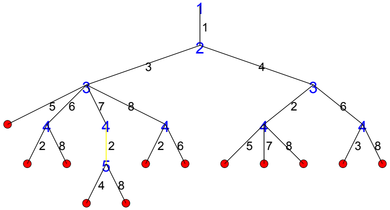


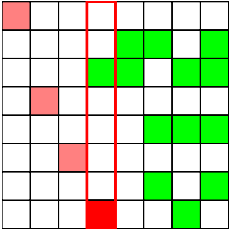


Back to Start
Skip Animation

Cork Constraint Computation Centre

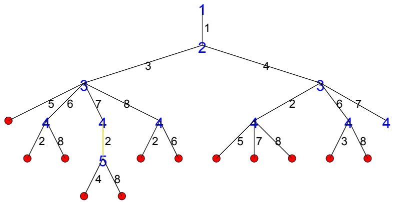
Default Strategy

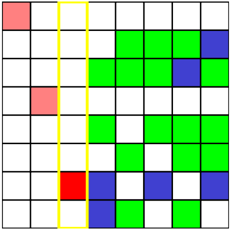




Back to Start
Skip Animation

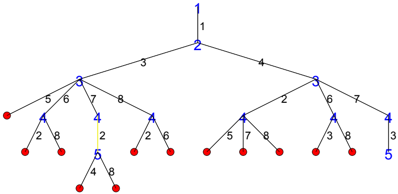
Default Strategy

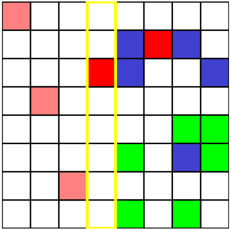




Back to Start
Skip Animation

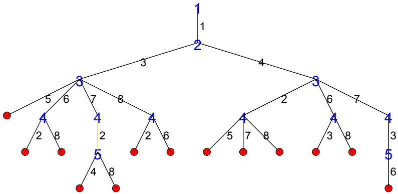
Default Strategy

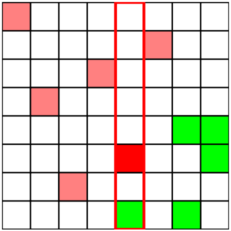




◂ Back to Start ▸▸ Skip Animation

Default Strategy





Back to Start
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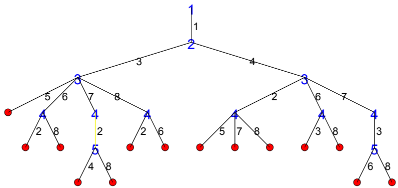

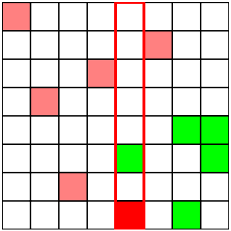

Back to Start
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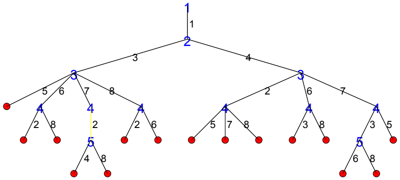

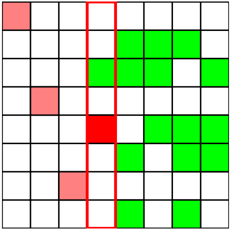

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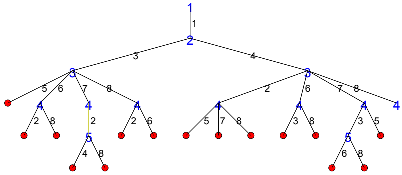

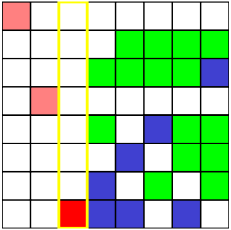

Back to Start
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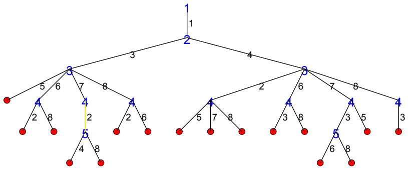

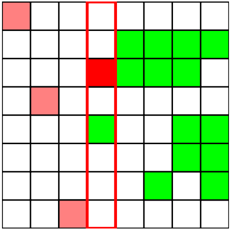

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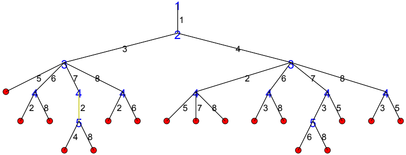
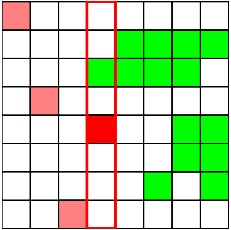
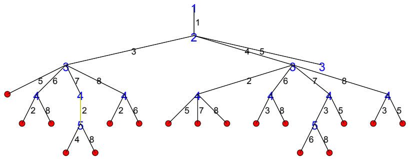


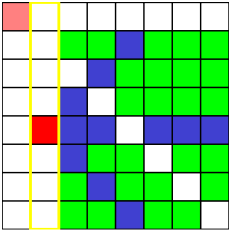


Back to Start
Skip Animation

Cork Constraint Computation Centre

Default Strategy





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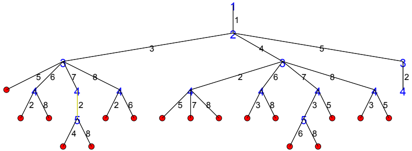

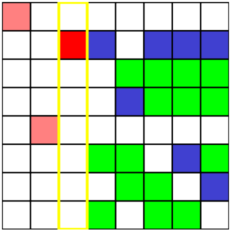

Back to Start
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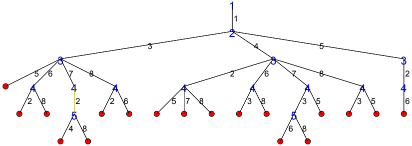

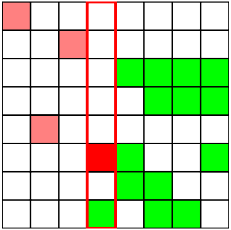

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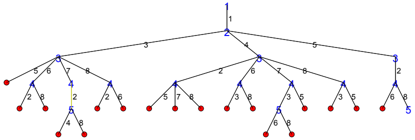
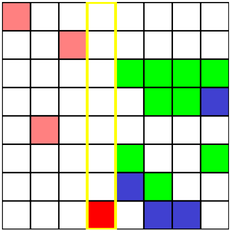
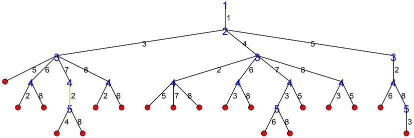
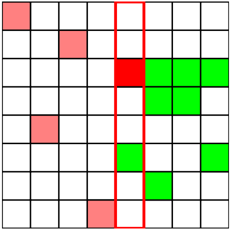




Back to Start   Skip Animation

Default Strategy





Back to Start
Skip Animation

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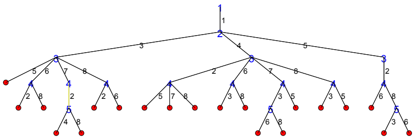

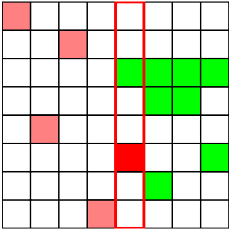

Back to Start
Skip Animation

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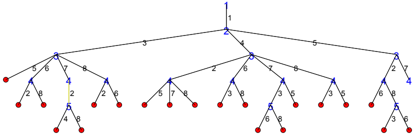

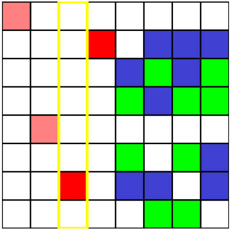

Back to Start
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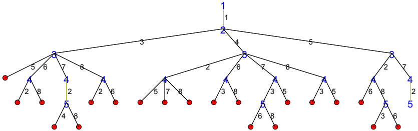
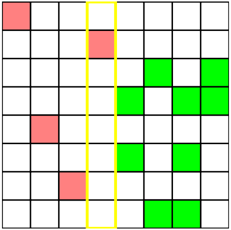
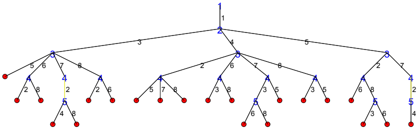
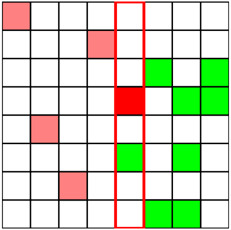




Back to Start   Skip Animation

Default Strategy





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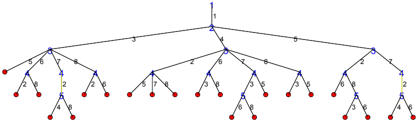

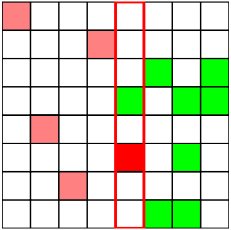

Back to Start   Skip Animation

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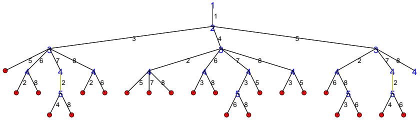

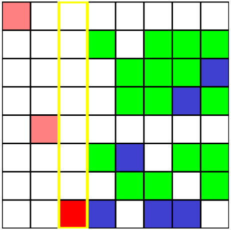

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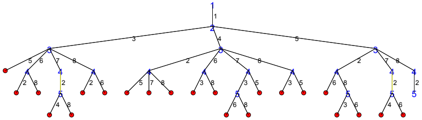

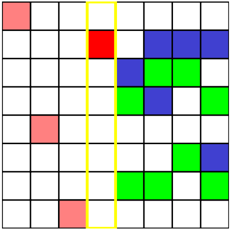

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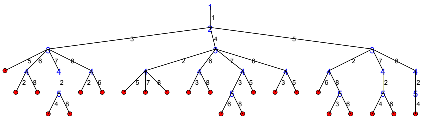

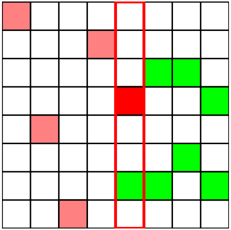

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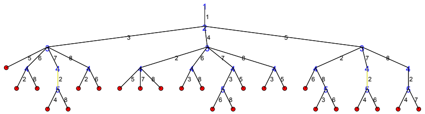

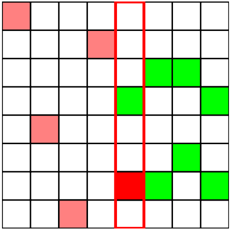

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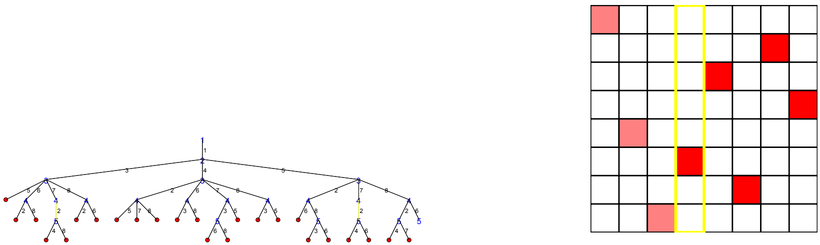

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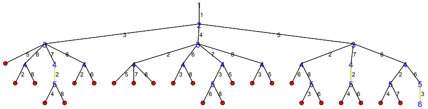
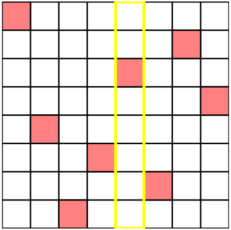
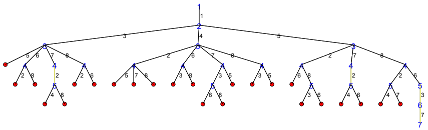
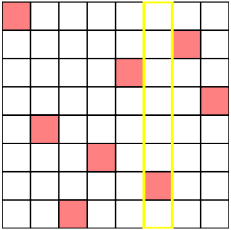




Back to Start | Skip Animation

Default Strategy





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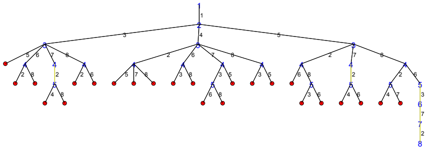

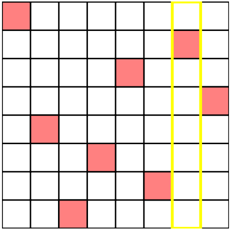

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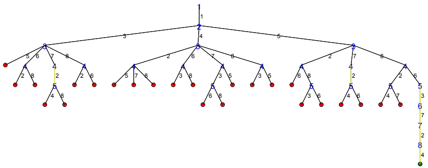
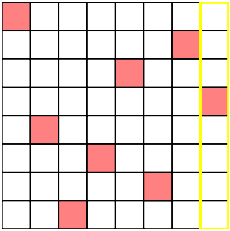
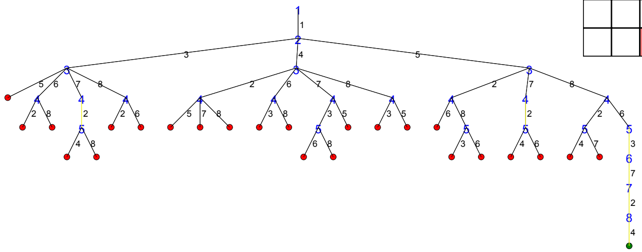
Default Strategy

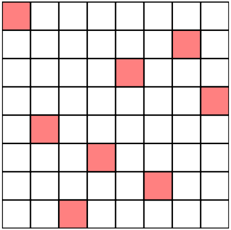




Back to Start

First Solution

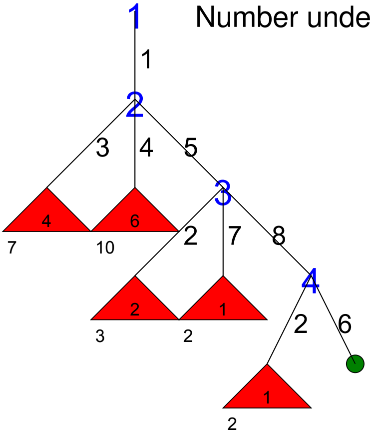




Observations

- Even for small problem size, tree can become large
- Not interested in all details
- Ignore all automatically fixed variables
- For more compact representation abstract failed sub-trees

Compact Representation

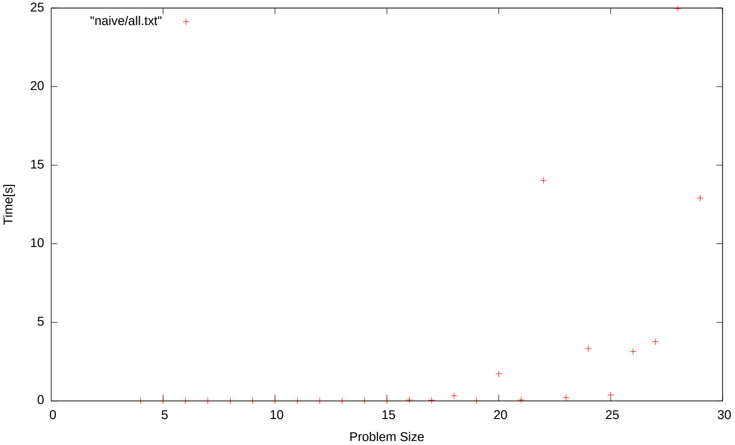


Number inside triangle:	Number of choices
Number under triangle:	Number of failures

Exploring other board sizes

- How stable is the model?
- Try all sizes from 4 to 100
- Timeout of 100 seconds

Naive Stategy, Problem Sizes 4-100



Observations

- Time very reasonable up to size 20
- Sizes 20-30 times very variable
- Not just linked to problem size
- No size greater than 30 solved within timeout

Outline

Possible Improvements

- Better constraint reasoning
  - Remodelling problem with 3 `alldifferent` constraints
  - Global reasoning as described before
  - Not explored here
- Better control of search
  - Static vs. dynamic variable ordering
  - Better value choice
  - Not using complete depth-first chronological backtracking

Static vs. Dynamic Variable Ordering

- Heuristic Static Ordering
  - Sort variables before search based on heuristic
  - Most important decisions
  - Smallest initial domain
- Dynamic variable ordering
  - Use information from constraint propagation
  - Different orders in different parts of search tree
  - Use all information available

First Fail strategy

- Dynamic variable ordering
- At each step, select variable with smallest domain
- Idea: If there is a solution, better chance of finding it
- Idea: If there is no solution, smaller number of alternatives
- Needs tie-breaking method

Modification of Program

```
:-module(nqueen).
:-export(top/0).
:-lib(ic).

top:-
    nqueen(8,L), writeln(L).

nqueen(N,L):-
    length(L,N),
    L :: 1..N,
    alldifferent(L),
    noattack(L),
    labeling(L).
```

⇨ *replace with*

Modification of Program

```
:-module(nqueen).
:-export(top/0).
:-lib(ic).

top:-
    nqueen(8,L), writeln(L).

nqueen(N,L):-
    length(L,N),
    L :: 1..N,
    alldifferent(L),
    noattack(L),
    search(L,0,first_fail,indomain,complete,[]).
```

Variable Choice

- Determines the order in which variables are assigned
- `input_order` assign variables in static order given
- `first_fail` select variable with smallest domain first
- `most_constrained` like first_fail, tie break based on number of constraints in which variable occurs
- Others, including programmed selection

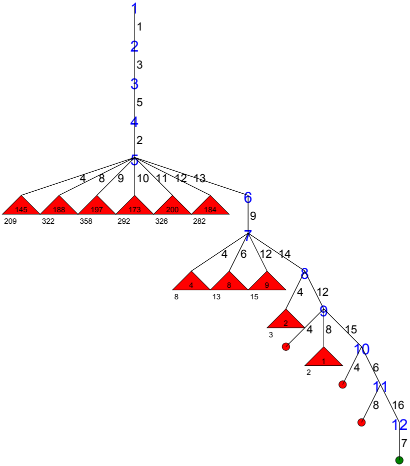
Value Choice

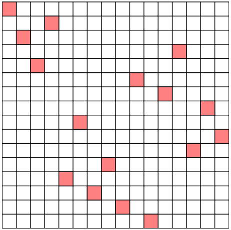
- Determines the order in which values are tested for selected variables
- `indomain` Start with smallest value, on backtracking try next larger value
- `indomain_max` Start with largest value
- `indomain_middle` Start with value closest to middle of domain
- `indomain_random` Choose values in random order

Comparison

- Board size 16x16
- Naive (Input Order) Strategy
- First Fail variable selection

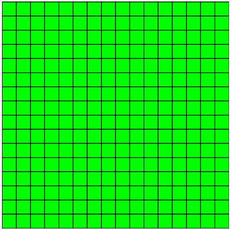
Naive (Input Order) Strategy (Size 16)





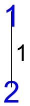
FirstFail Strategy (Size 16)

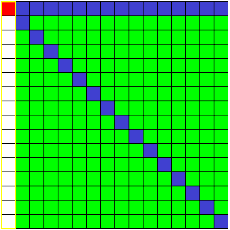
1



Skip Animation

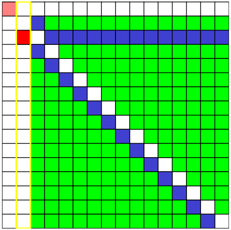
FirstFail Strategy (Size 16)





Back to Start
Skip Animation

FirstFail Strategy (Size 16)



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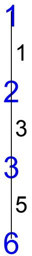
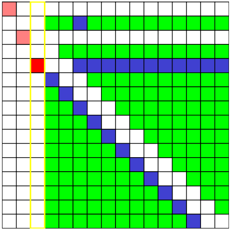
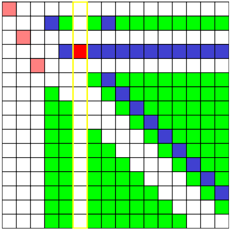
FirstFail Strategy (Size 16)





Back to Start   Skip Animation

FirstFail Strategy (Size 16)



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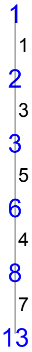
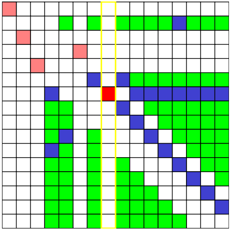
FirstFail Strategy (Size 16)





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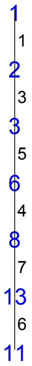
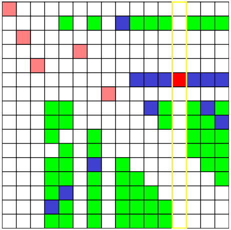
FirstFail Strategy (Size 16)





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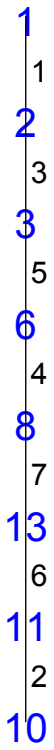
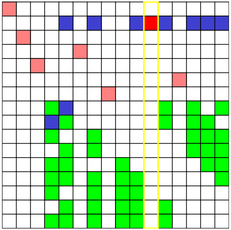
FirstFail Strategy (Size 16)





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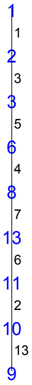

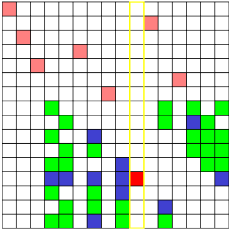

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Skip Animation

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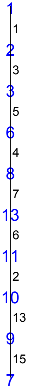

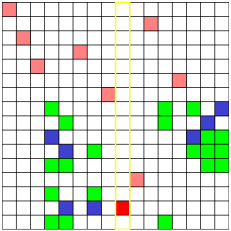

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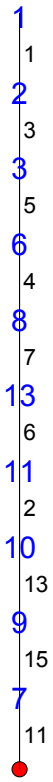

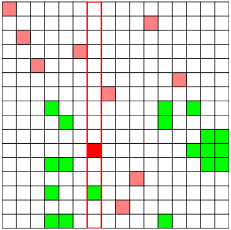

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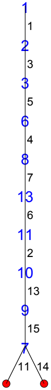

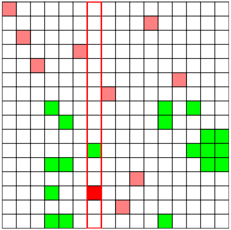

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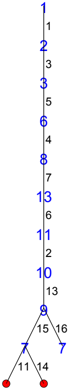

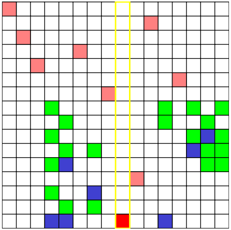

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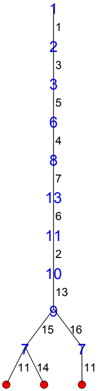

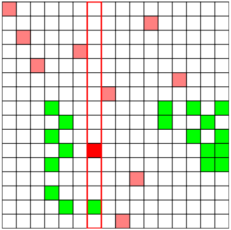

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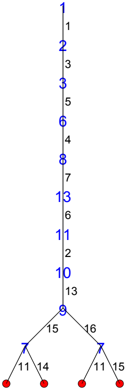

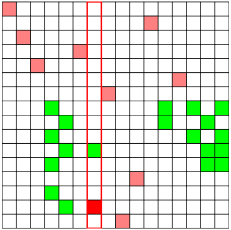

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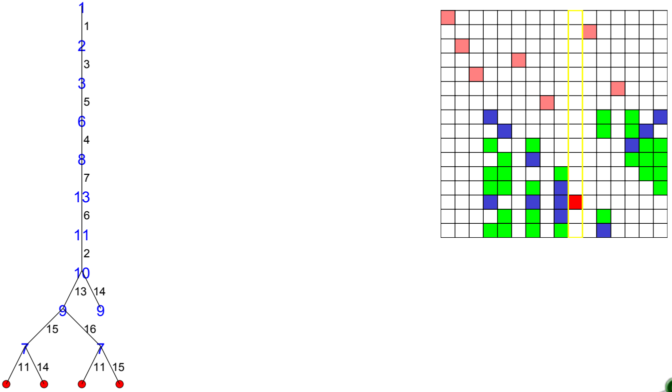

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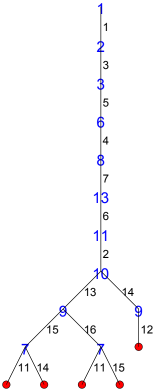

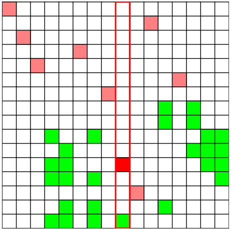

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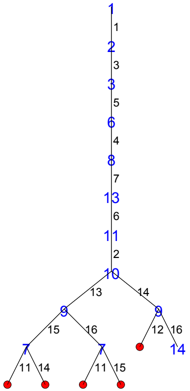

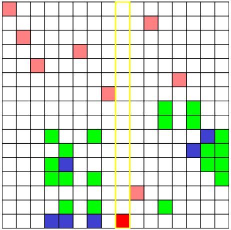

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Skip Animation

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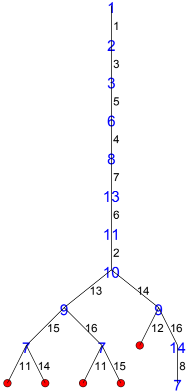

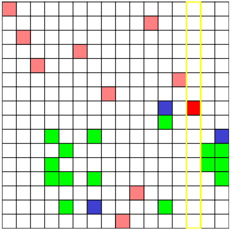

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Skip Animation

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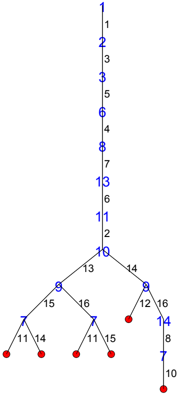

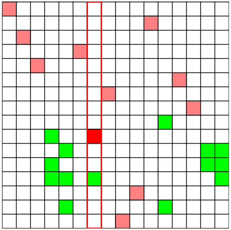

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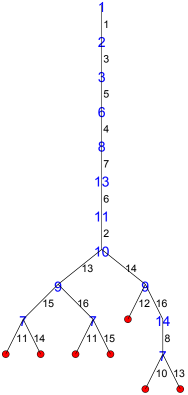
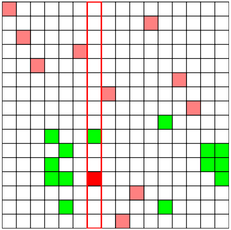
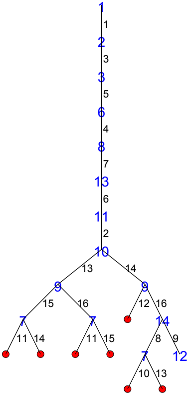


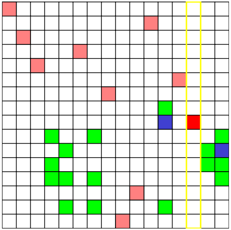


Back to Start
Skip Animation

Cork Constraint Computation Centre

FirstFail Strategy (Size 16)





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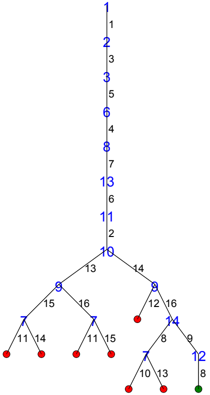
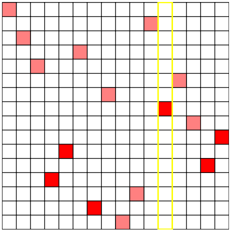
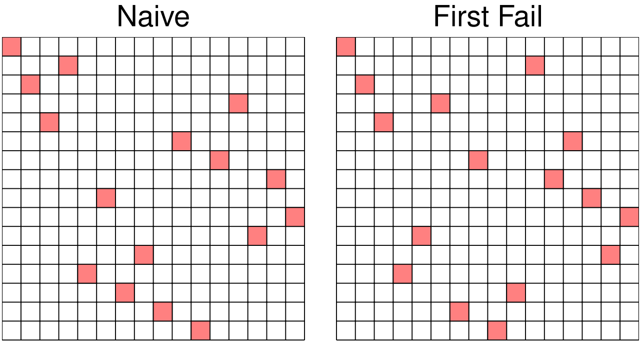
Skip Animation

FirstFail Strategy (Size 16)



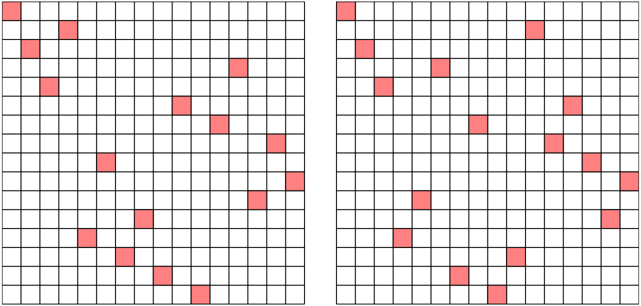


Comparing Solutions



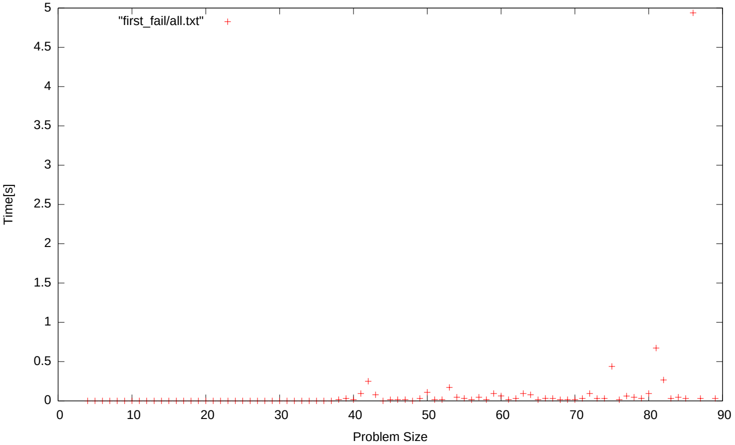
Comparing Solutions

Naive | First Fail



Solutions are different!

FirstFail, Problem Sizes 4-100



Observations

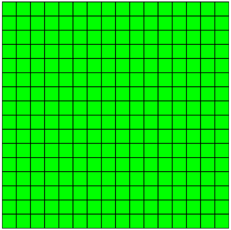
- This is much better
- But some sizes are much harder
- Timeout for sizes 88, 91, 93, 97, 98, 99

Can we do better?

- Improved initial ordering
  - Queens on edges of board are easier to assign
  - Do hard assignment first, keep simple choices for later
  - Begin assignment in middle of board
- Matching value choice
  - Values in the middle of board have higher impact
  - Assign these early at top of search tree
  - Use `indomain_middle` for this

Start from Middle (Size 16)

9



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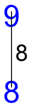

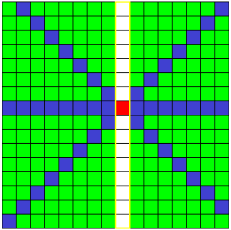

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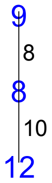

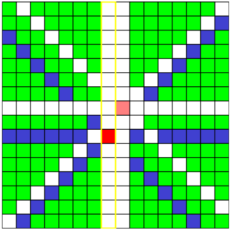

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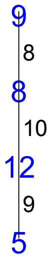

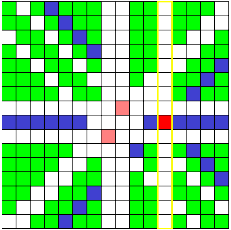

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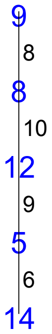

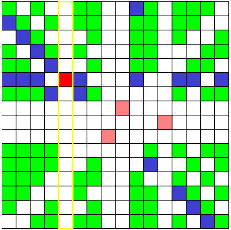

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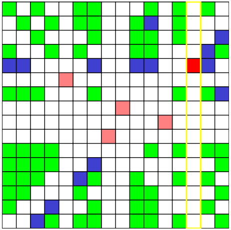

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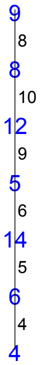

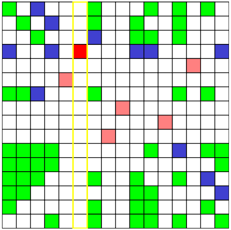

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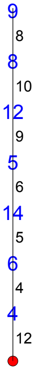

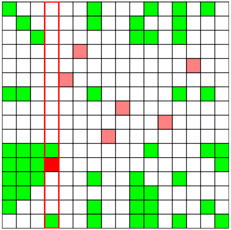

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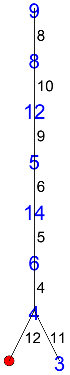

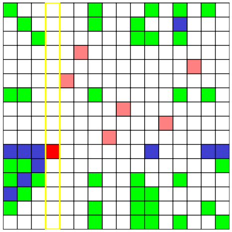

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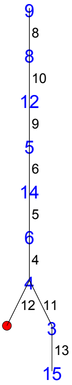

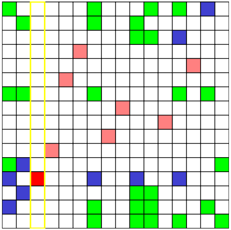

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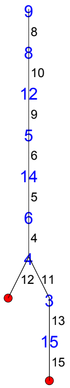

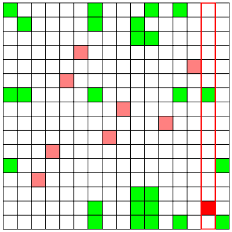

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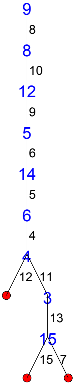

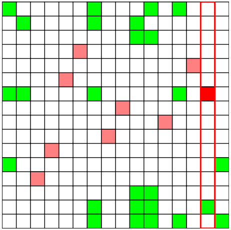

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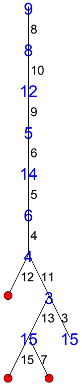

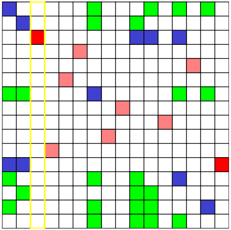

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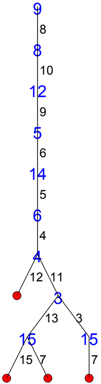

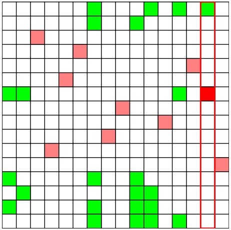

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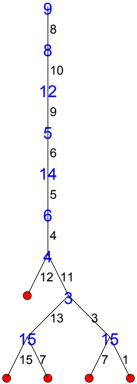

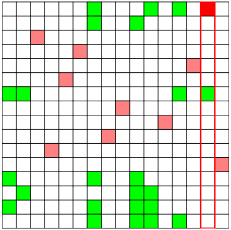

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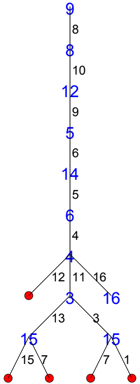

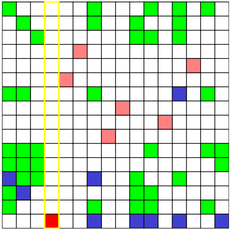

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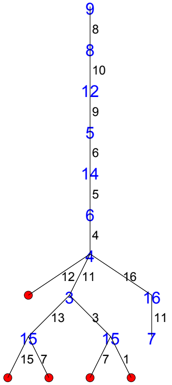

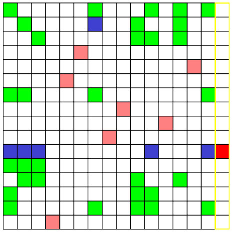

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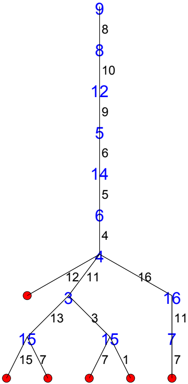

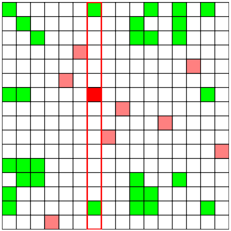

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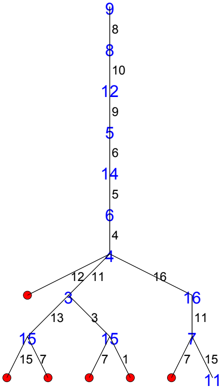

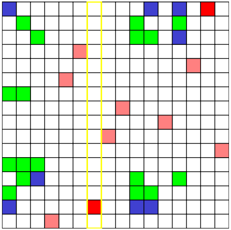

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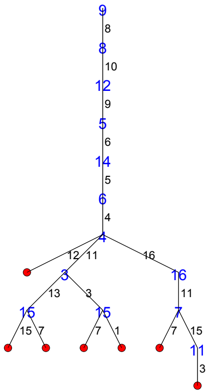

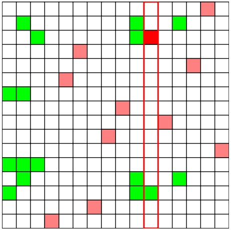

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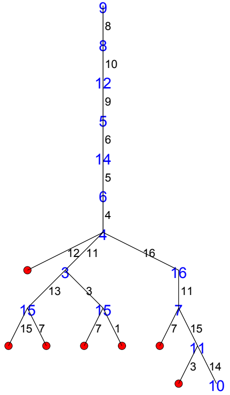

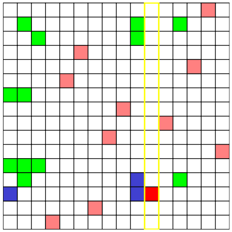

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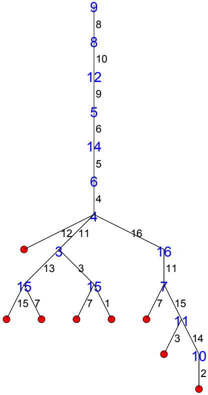

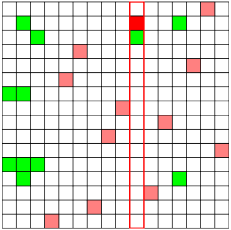

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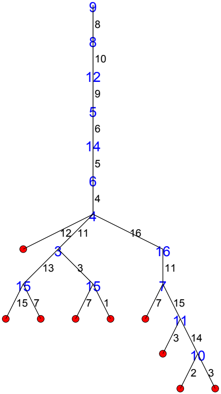

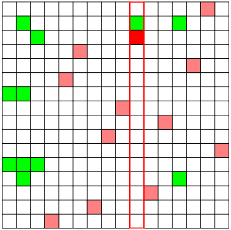

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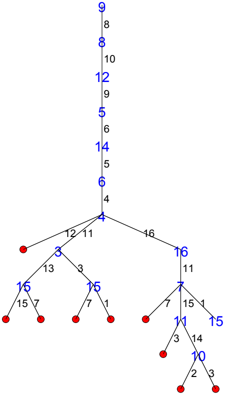

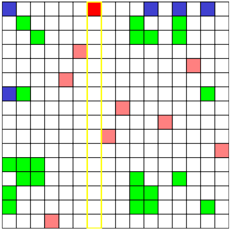

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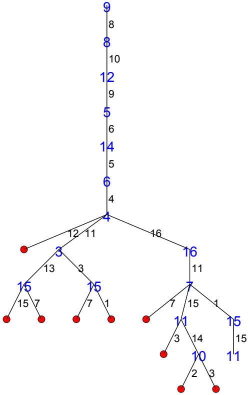

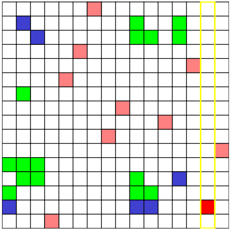

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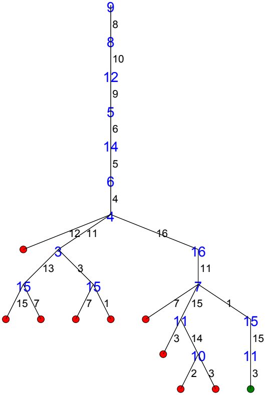
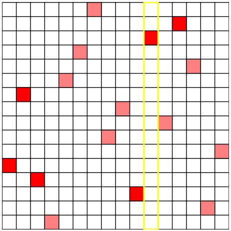
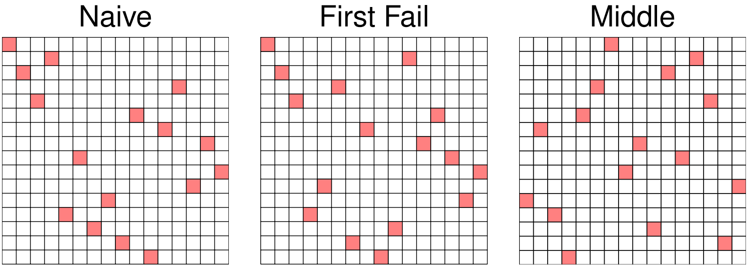


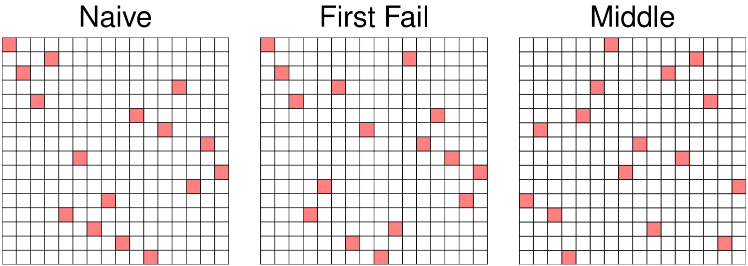


Back to Start

Comparing Solutions

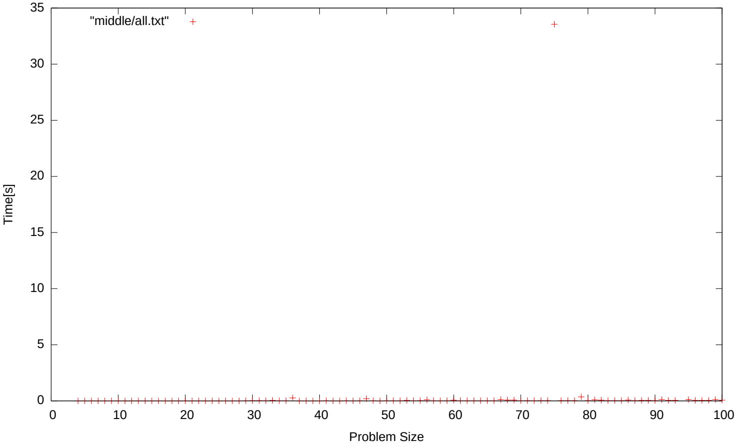


Comparing Solutions



Again, solutions are different!

Middle, Problem Sizes 4-100



Observations

- Not always better than first fail
- For size 16, trees are similar size
- Timeout only for size 94
- But still, one strategy does not work for all problem sizes
- There are ways to resolve this!

Approach 1: Heuristic Portfolios

- Try multiple strategies for the same problem
- With multi-core CPUs, run them in parallel
- Only one needs to be successful for each problem

Approach 2: Restart with Randomization

- Only spend limited number of backtracks for a search attempt
- When this limit is exceeded, restart at beginning
- Requires randomization to explore new search branch
- Randomize variable choice by random tie break
- Randomize value choice by shuffling values
- Needs strategy when to restart

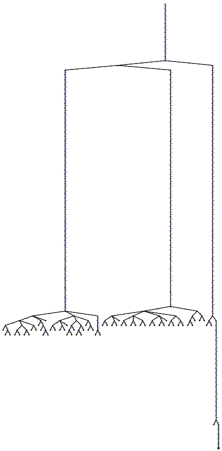
Approach 3: Partial Search

- Abandon depth-first, chronological backtracking
- Don't get locked into a failed sub-tree
- A wrong decision at a level is not detected, and we have to explore the complete subtree below to undo that wrong choice
- Explore more of the search tree
- Spend time in promising parts of tree

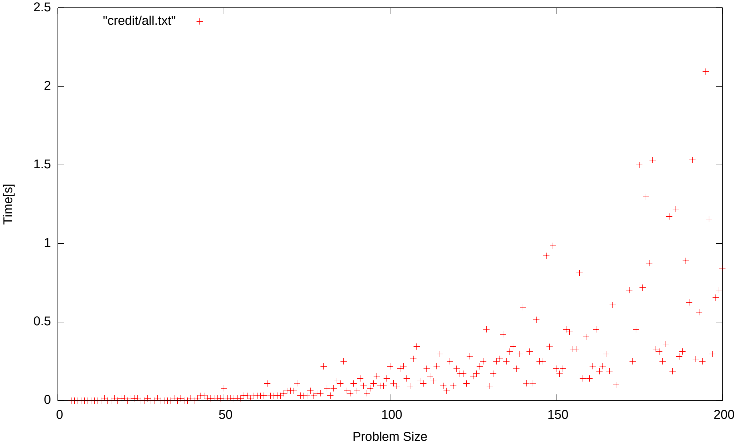
Example: Credit Search

- Explore top of tree completely, based on credit
- Start with fixed amount of credit
- Each node consumes one credit unit
- Split remaining credit amongst children
- When credit runs out, start bounded backtrack search
- Each branch can use only K backtracks
- If this limit is exceeded, jump to unexplored top of tree

Credit, Search Tree Problem Size 94



Credit, Problem Sizes 4-200



Points to Remember

- Choice of search can have huge impact on performance
- Dynamic variable selection can lead to large reduction of search space
- Packaged search can do a lot, but programming search adds even more
- Depth-first chronologicial backtracking not always best choice
- How to control this explosion of search alternatives?

Part IV

What is missing?

Many Specialized Topics

- How to design efficient core engine
- Hybrids with LP/MIP tools
- Hybrids with SAT
- Symmetry breaking
- Use of MDD/BDD to encode sets of solutions
- High level modelling tools
- Debugging/visualization

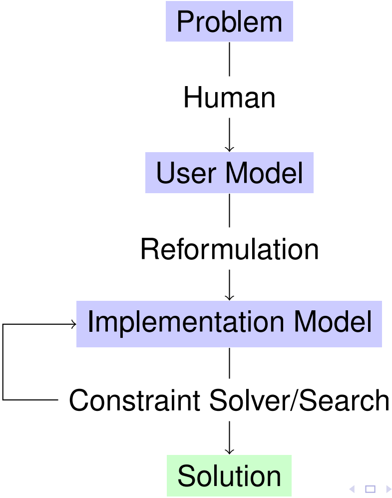
Reformulation

- Just because the user has modelled it this way, it doesn't mean we have to solve it that way
  - Replace some constraint(s) by other, equivalent constraints
  - Because we don't have that constraint in our system
  - For performance

Learning

- While solving the problem we can learn how to strengthen the model/search
  - Understand which constraints/method contribute to propagation and change schedule
  - Learn no-good constraints by explaining failure
  - Adapt search strategy based on search experience

Refined Process



What is CP actually used for?

`http://hsimonis.wordpress.com`