Classical and Non-Classical Uses of SAT in Model-Checking

Jean-François Raskin
Université Libre de Bruxelles

Objectives

- Give representative **examples** of the use of SAT solvers in **verification algorithms** for finite state systems
- **Disclaimer I**: not my work
- **Disclaimer II**: by no means a full review of the literature (examples only)

Plan

- Bounded model-checking
- Unbounded model-checking
- Inductive invariant generation

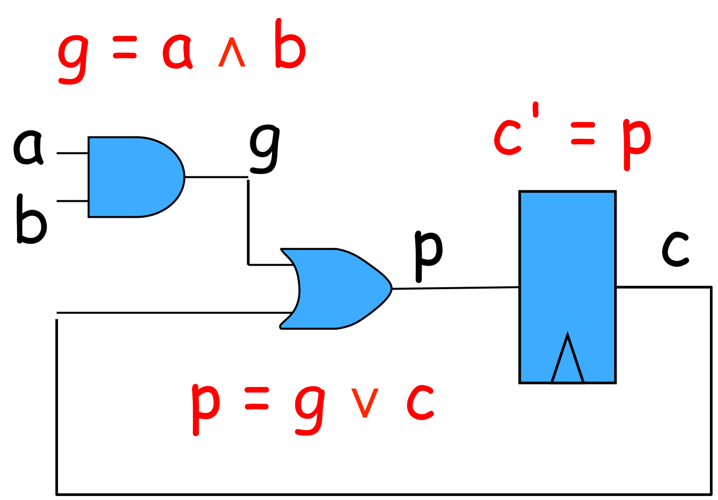
Symbolic transition systems

- A Symbolic Transition System (STS) S=(X,I,T) where:
  - X is a set of boolean variables
  - $I \in \mathfrak{B}(X)$ defines the initial states
  - $T \in \mathfrak{B}(X \cup X')$ defines the transition relation
- We associate to STS=(X,I,T) an explicit, so exponentially larger, transition system TS=(S,S_0,E):
  - $S = \{ v \mid v : X \to \{0,1\} \}$
  - $S_0 = \{ v \in S \mid v \models I \}$
  - $E = \{ (v,v') \mid (v,v') \models T \}$

Typical verification questions

- **Safety**: are all the executions of my system avoiding a set of bad states ?
- **Reachability**: is there an execution of my system that reaches bad states ? dual of safety
- **Liveness**: are all the executions of my system doing eventually/repeatedly something good ?

Circuit Example



From McMillan03

Model:

$C = \{$
$g = a \wedge b,$
$p = g \vee c,$
$c' = p$
$\}$

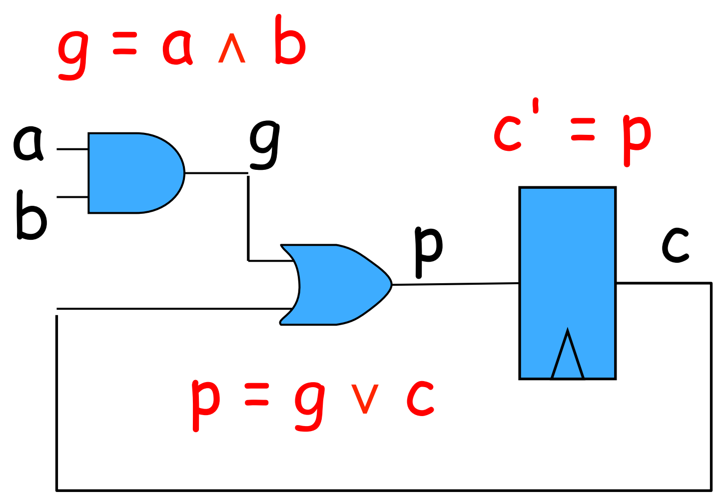
Can we reach a state of the circuit in which $c \wedge \neg p$ holds ?

Bounded model-checking [BCC+99]

Bounded model-checking

- First, let us **falsifying safety properties**
- Let STS=(X,I,T) and Bad $\in \mathfrak{B}$(X)
- Is there a ⟦T⟧-path from ⟦I⟧ to ⟦Bad⟧ ?
- **Bound**: Is there a ⟦T⟧-**path of length at most k** from ⟦I⟧ to ⟦Bad⟧ ?

System unfolding



Model:

$C = \{$
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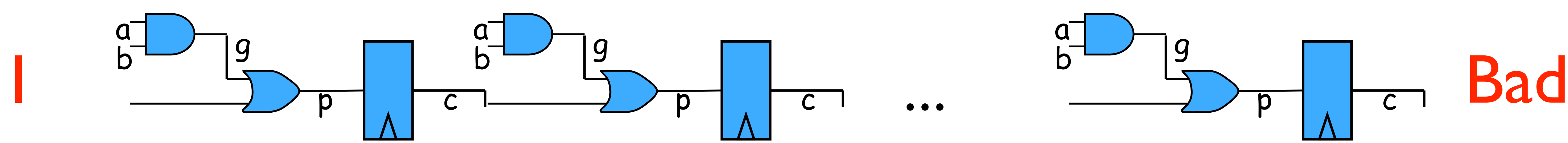
k unfolding



Can the circuit reach a state where c is true in at most k steps ?

Unfolding of T

- **Unfolding** of T k times:

  $T(X_0,X_1) \wedge T(X_1,X_2) \wedge ... \wedge T(X_{k-2},X_{k-1})$

- Use SAT solver to check **satisfiability** of

  $I(X_0) \wedge T(X_0,X_1) \wedge T(X_1,X_2) \wedge ... \wedge T(X_{k-2},X_{k-1}) \wedge \bigvee_{i=0..k-1} Bad(X_i)$

- A satisfying assignment corresponds to a path of length at most k from $[\![I]\!]$ to $[\![Bad]\!]$, i.e. a **counter-example** to the safety property

Beyond safety

- Let Good $\in \mathfrak{B}(x)$
- Given an infinite path ρ in TS, we note **Inf**(ρ) the set of states that appear infinitely many times along ρ
- An infinite path in TS is *good* if $\mathrm{Inf}(\rho) \cap [\![\mathrm{Good}]\!] \neq \emptyset$
- **Liveness**: check that every path in TS are *good*
- Counter-examples are **lasso-path** such that the cycle does not contain any good states
- **Bound**: find a lasso-path of length at most k that does not cross $[\![\mathrm{Good}]\!]$ in the lasso part

Beyond safety

- Encoding in SAT:

$I(X_0)$
$\wedge T(X_0,X_1) \wedge \ldots \wedge T(X_{k-2},X_{k-1})$
$\wedge \bigvee_{m=0..k-1}$ $T(X_{k-1},X_m)$
$\wedge_{j=m..k-1} \neg Good(Xj)$

Lasso

Liveness is violated

Beyond counter-examples

- **Proving properties** is only possible if k is taken sufficiently large
- **Diameter**: maximum length of the shortest path between any two states
- ... is worst-case exponential, furthermore it is PSpace-C to compute it
- So, other techniques are needed

Unbounded Model-Checking

Four examples of unbounded SAT based MC

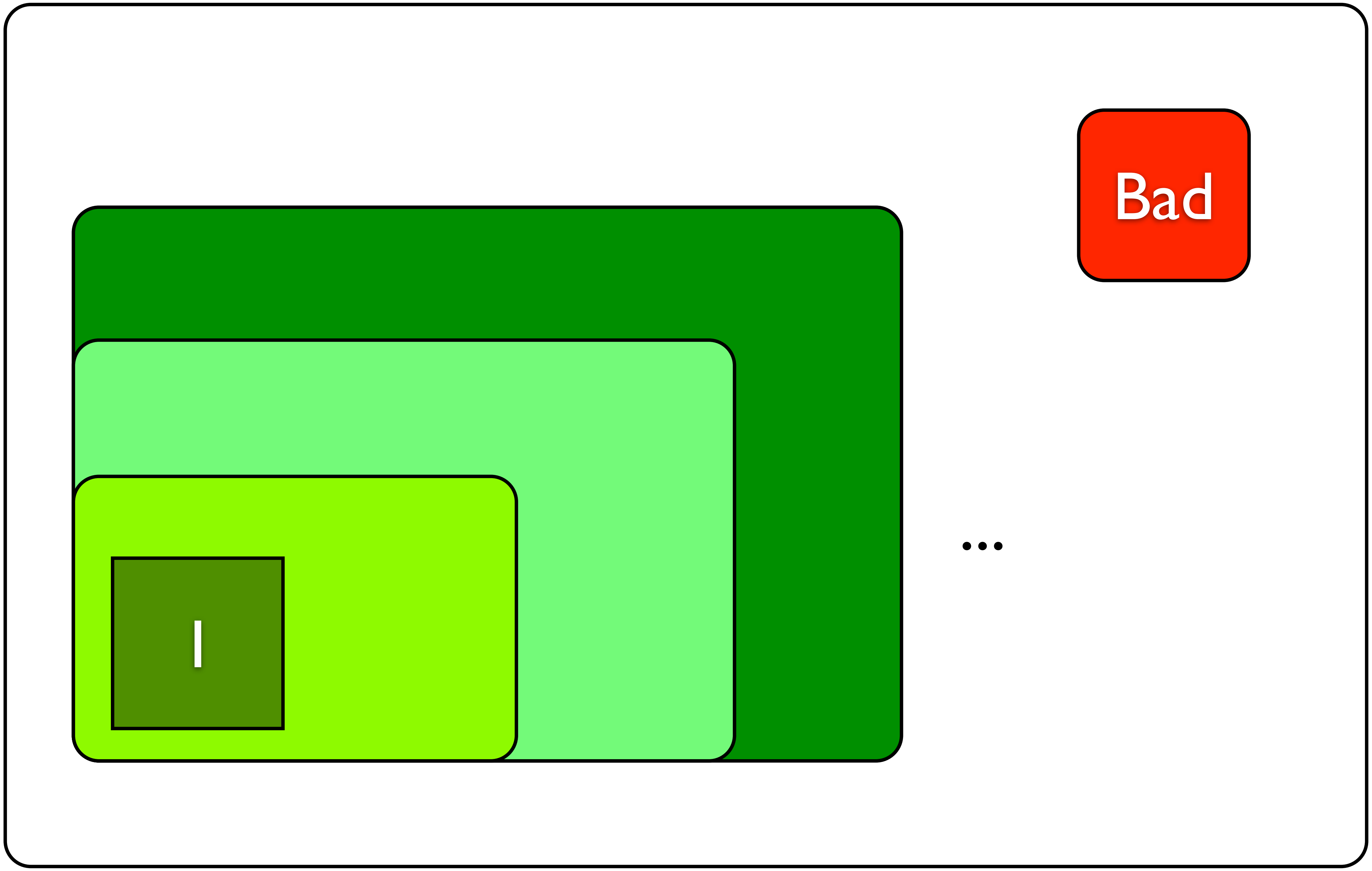
- Symbolic Reachability Analysis based on SAT Solvers [ABE00]
- Unbounded Sat-based model-checking with abstractions [CCKSVW02] + McMillan variant
- Interpolation and unbounded SAT-based model-checking [McMillan03]
- Discovering inductive invariants in subset constructions

Symbolic Reachability Analysis based on SAT Solvers [ABE00]

Symbolic Forward/Backward Reachability

- Let STS=(X,I,T) and let Bad ∈ $\mathfrak{B}$(X)
- **ReachFwd**(I) is the least set of states R such that R=⟦I⟧∪Post⟦T⟧(R)

Forward exploration



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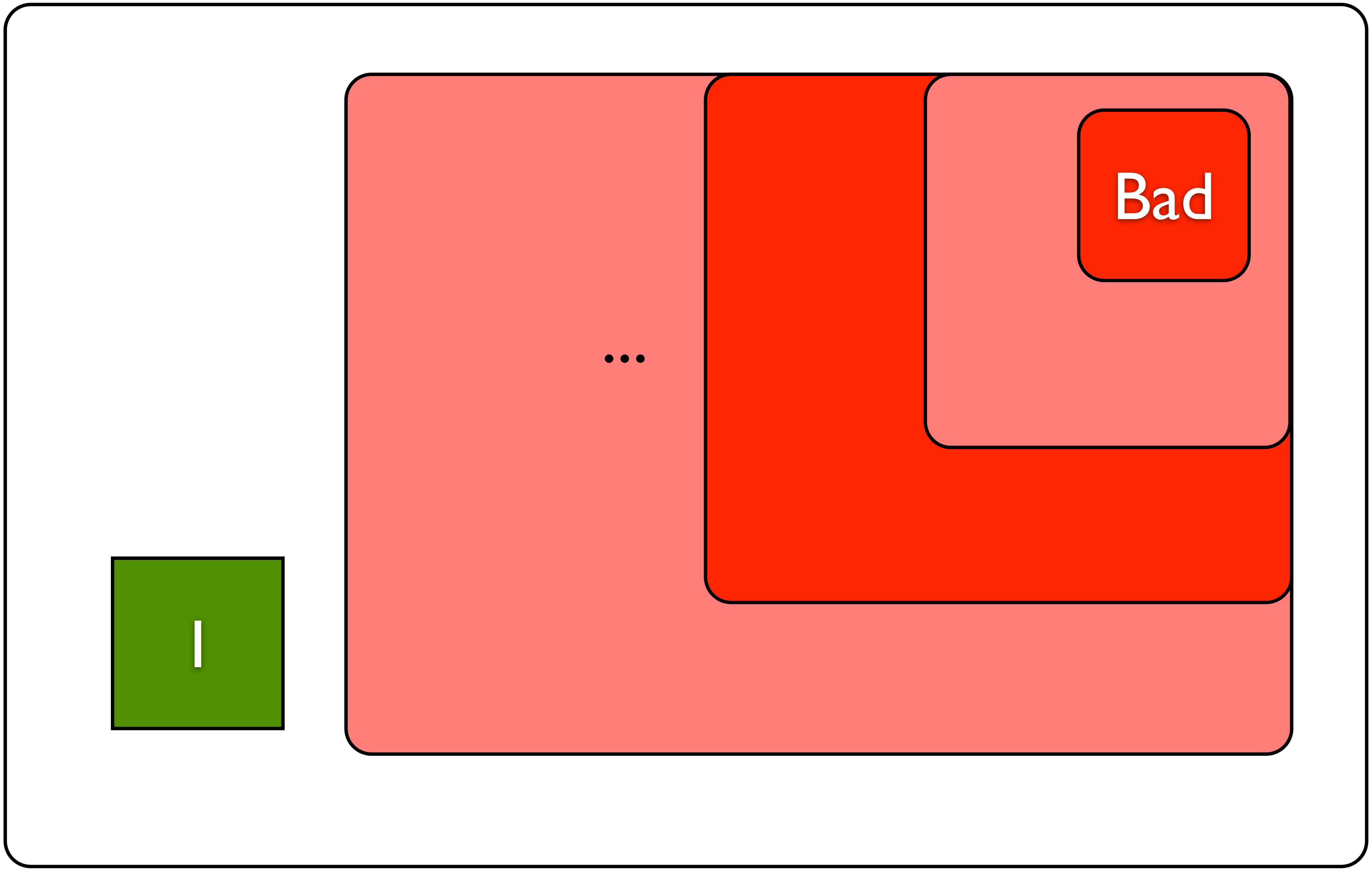
- **ReachBack**(Bad) is the least set of states B such that B=⟦Bad⟧∪Pre⟦T⟧(B)

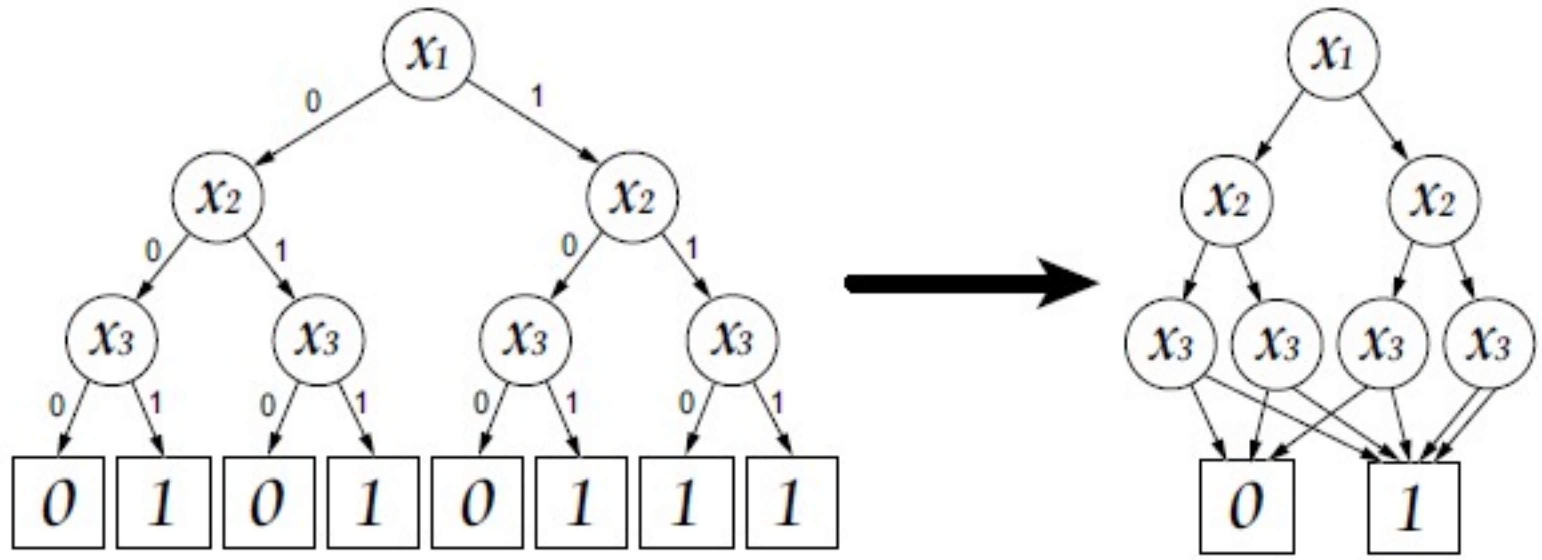
Forward exploration



Symbolic Forward/Backward Reachability

- Let STS=(X,I,T) and let Bad $\in \mathfrak{B}(X)$
- **ReachFwd**(I) is the least set of states R such that R=$[\![I]\!] \cup$Post$[\![T]\!]$(R)
- **ReachBack**(Bad) is the least set of states B such that B=$[\![Bad]\!] \cup$Pre$[\![T]\!]$(B)
- Symbolic MC: fixpoints+**data structure** for manipulating sets

BDDs



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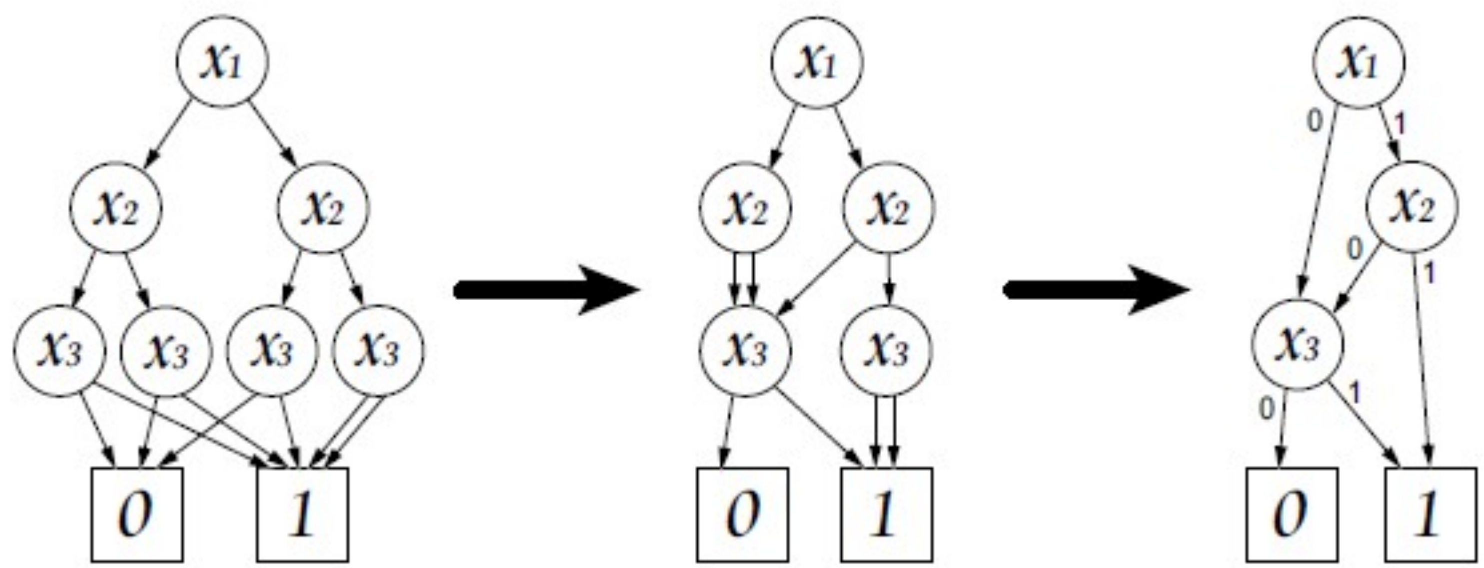

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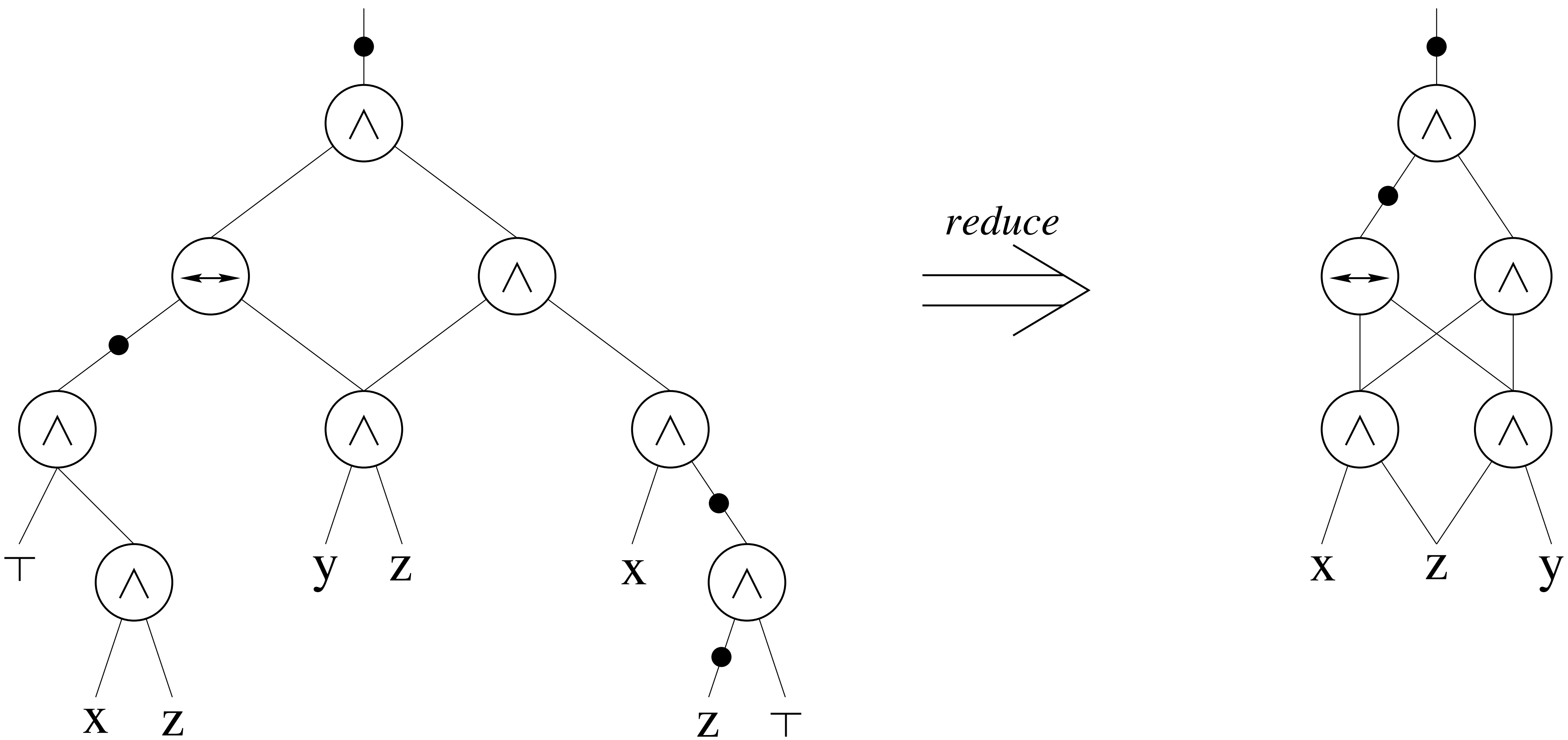
BDDs - Canonicity and Succinctness

- BDDs are **canonical** representation for Boolean functions
- Make very **easy** to check fixed-point
- Fact: some Boolean functions have **provably large** BDD representations, e.g. binary multiplication
- **Idea**: use potentially more compact representations... at the expense of **canonicity** and (maybe) some algorithmic efficiency

Boolean circuits



Boolean circuits

- As BDDs, **Boolean circuits** represent sets of valuations (=states)
- There is **no** (useful) canonical form
- There are often **more compact** than BDDs
- Algorithms for constructing new BCs from existing ones
-

Boolean circuits and existential quantification

- Expansion rule

$$\exists x\ .\ \phi(x) \iff \phi(\bot) \vee \phi(\top)$$

- To avoid blow-up:

Inlining:

$$\exists x\ .\ (x \leftrightarrow \psi) \wedge \phi(x) \iff \phi(\psi) \qquad \text{(where } x \notin \text{Vars}(\psi))$$

Scope Reduction:

$$\exists x\ .\ \phi(x) \wedge \psi \iff (\exists x . \phi(x)) \wedge \psi \qquad \text{(where } x \notin \text{Vars}(\psi))$$

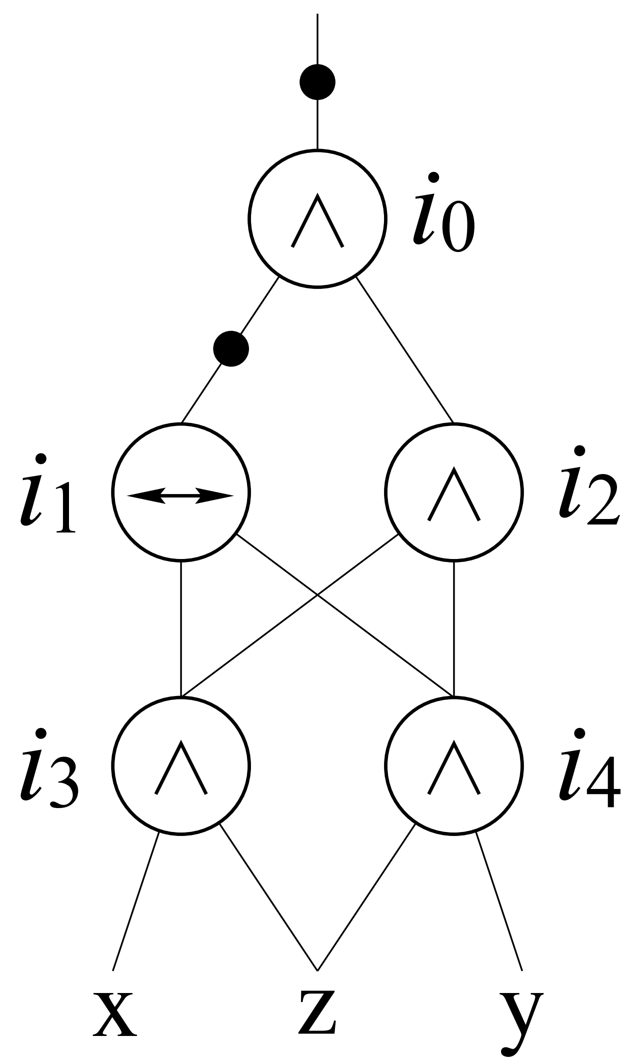
$$\exists x\ .\ \phi(x) \vee \psi(x) \iff (\exists x . \phi(x)) \vee (\exists x . \psi(x))$$

Boolean circuits

- As BDDs, Boolean circuits represent sets of valuations
- There is **no** (useful) canonical form
- There are often **more compact** than BDDs
- Algorithms for constructing new BCs from existing ones
- Satisfiability is **NP-Complete**

→ **use SAT**

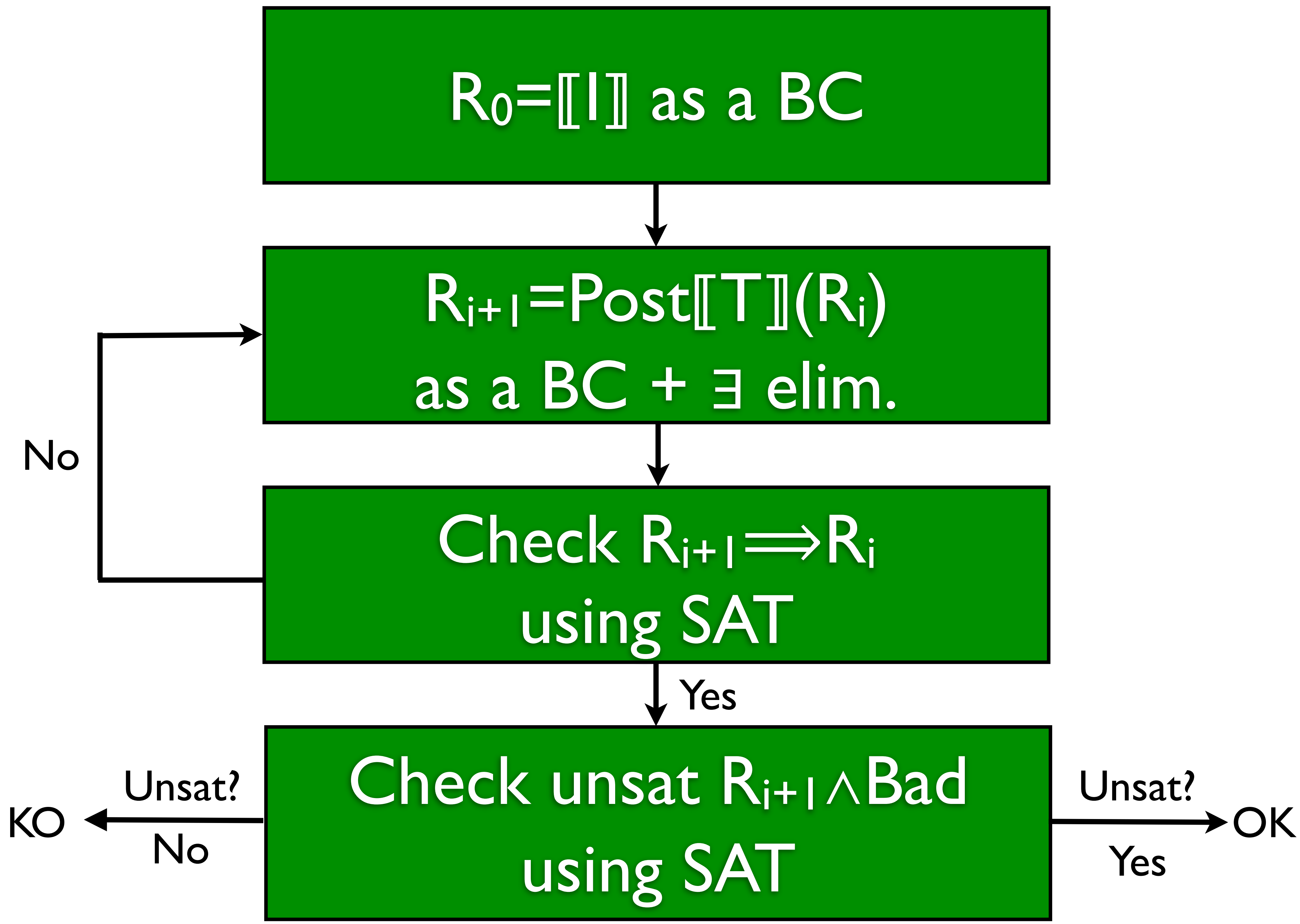
Checking satisfiability of Boolean circuits with SAT



$$
\begin{aligned}
&(i_0 \leftrightarrow \neg i_1 \wedge \ i_2) \\
\wedge\ &(i_1 \leftrightarrow i_3 \leftrightarrow i_4) \\
\wedge\ &(i_2 \leftrightarrow i_3 \wedge i_4) \\
\wedge\ &(i_3 \leftrightarrow x \wedge z) \\
\wedge\ &(i_4 \leftrightarrow z \wedge y) \\
\wedge\ &\neg i_0
\end{aligned}
$$

Not equivalent but **satisfiability** is maintained

SMC algorithm using BC and SAT

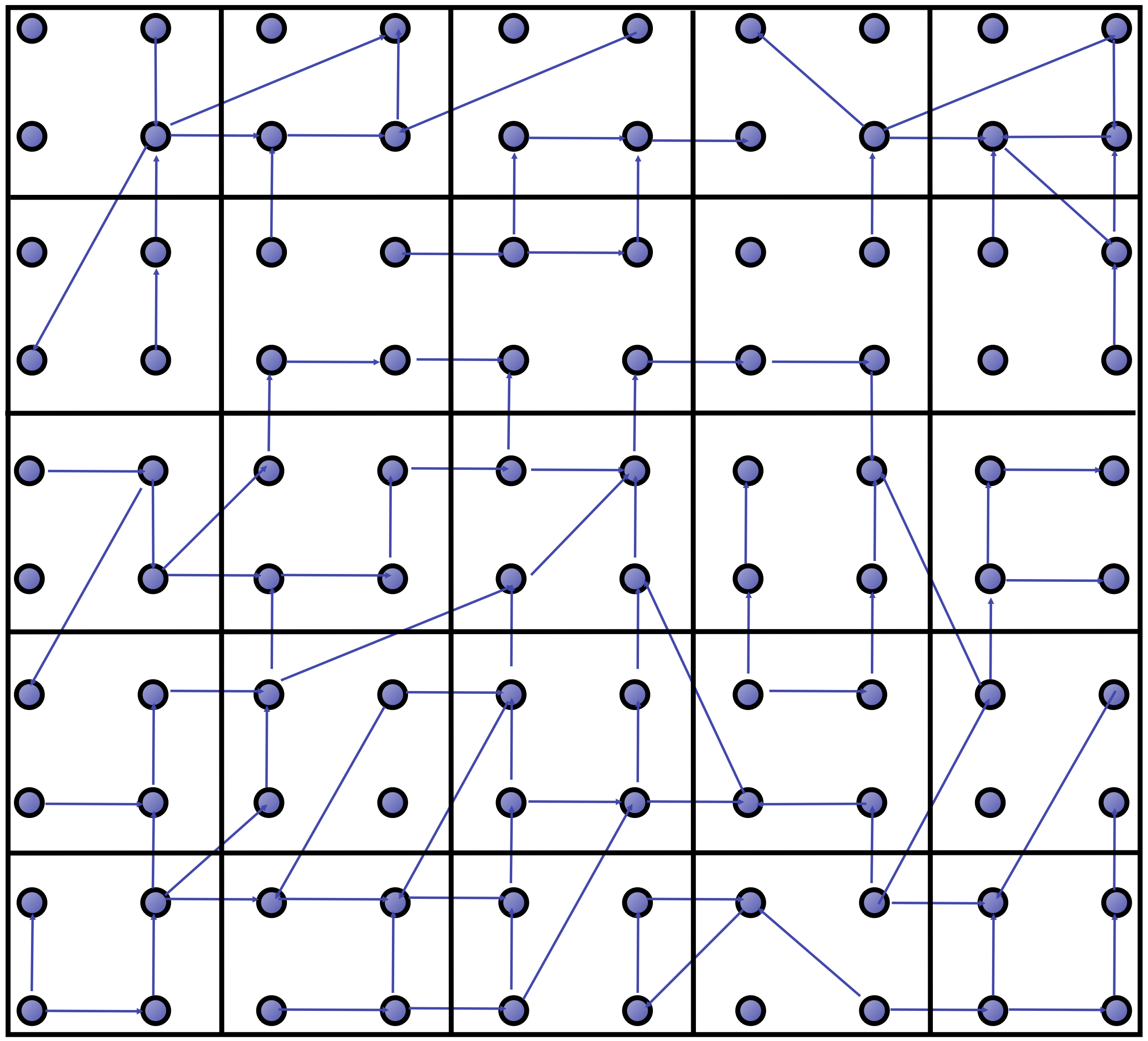


Unbounded SAT-based model-checking with abstractions [CCKSVW02]

Abstractions

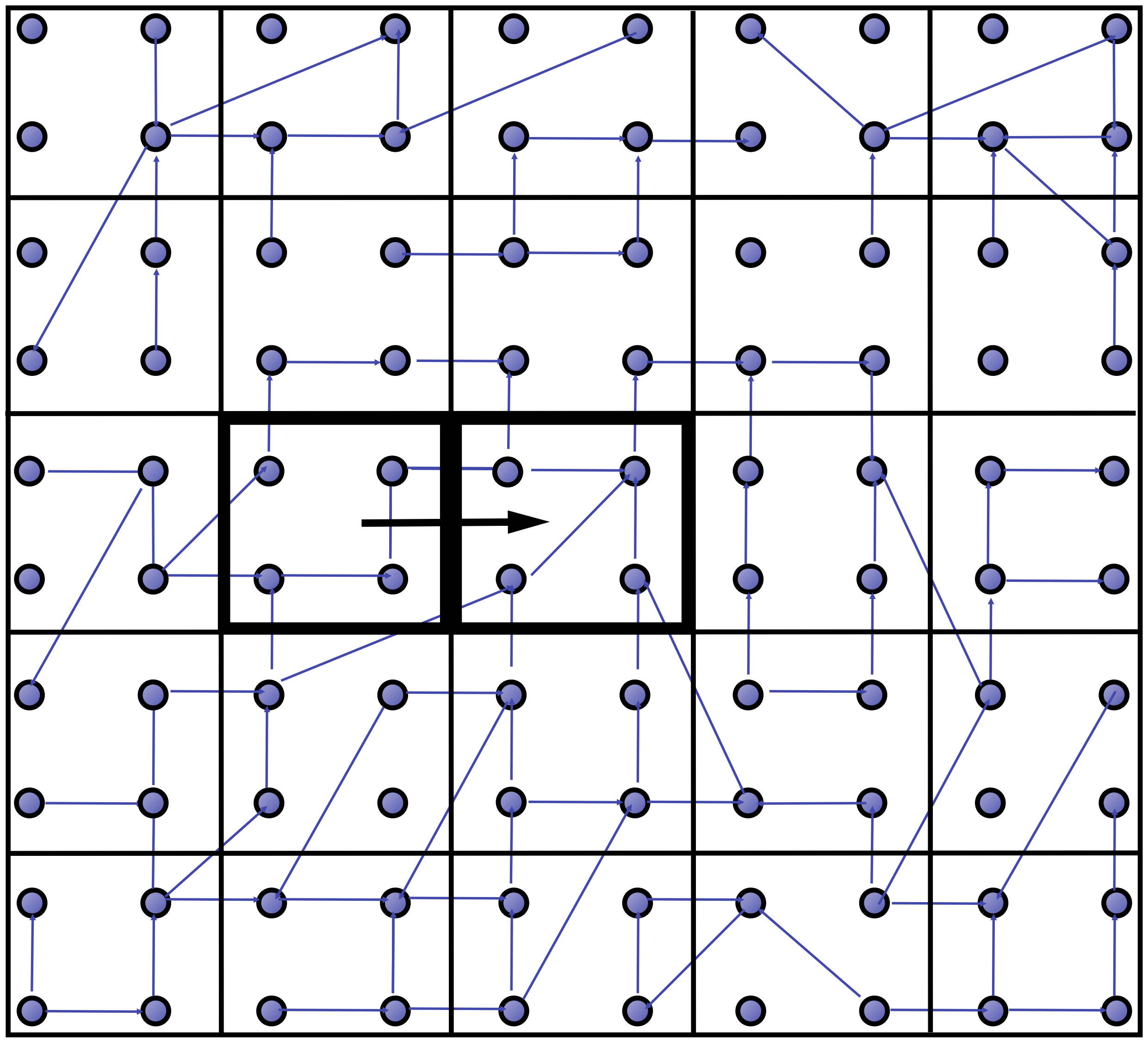
- Symbolic model-checking sensitive to the **number** of Boolean variables (symbolic state explosion problem)
- But (coarse) abstractions are often **sufficient** to prove correctness
- Try to **lower the number of variables** using abstraction

State-space partitioning



- ☛ **Predicates** on program/circuit state space
- ☛ States satisfying same predicates are **equivalent**
- ☛ Merged into one **abstract** state

State-space partitioning



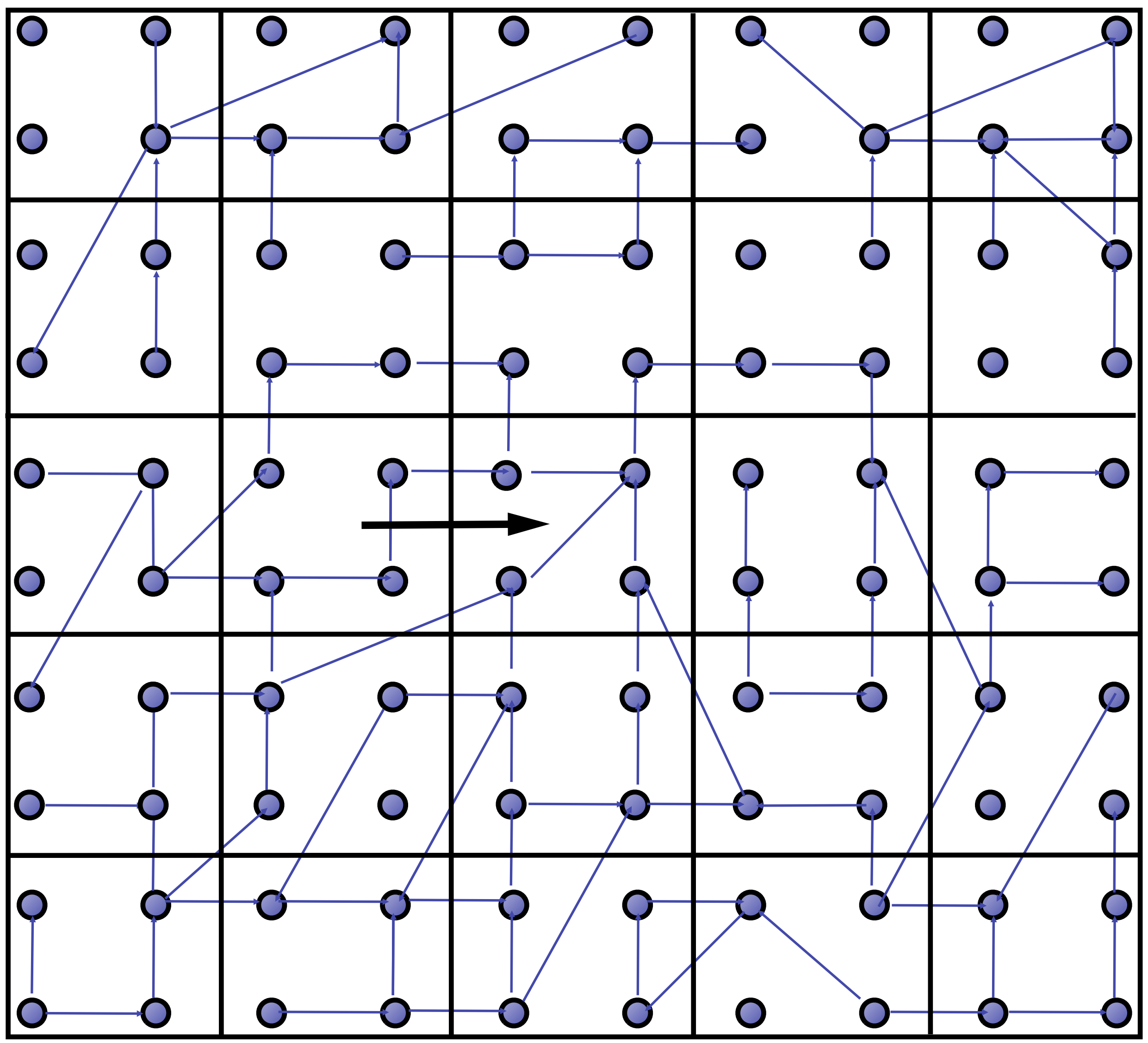
Abstract transition relation

$T^{\alpha}(A_1,A_2)$

iff

$\exists_{s_1 \in A_1} \cdot \exists_{s_2 \in A_2} \cdot T(s_1,s_2)$

State-space partitioning



Existential Lifting

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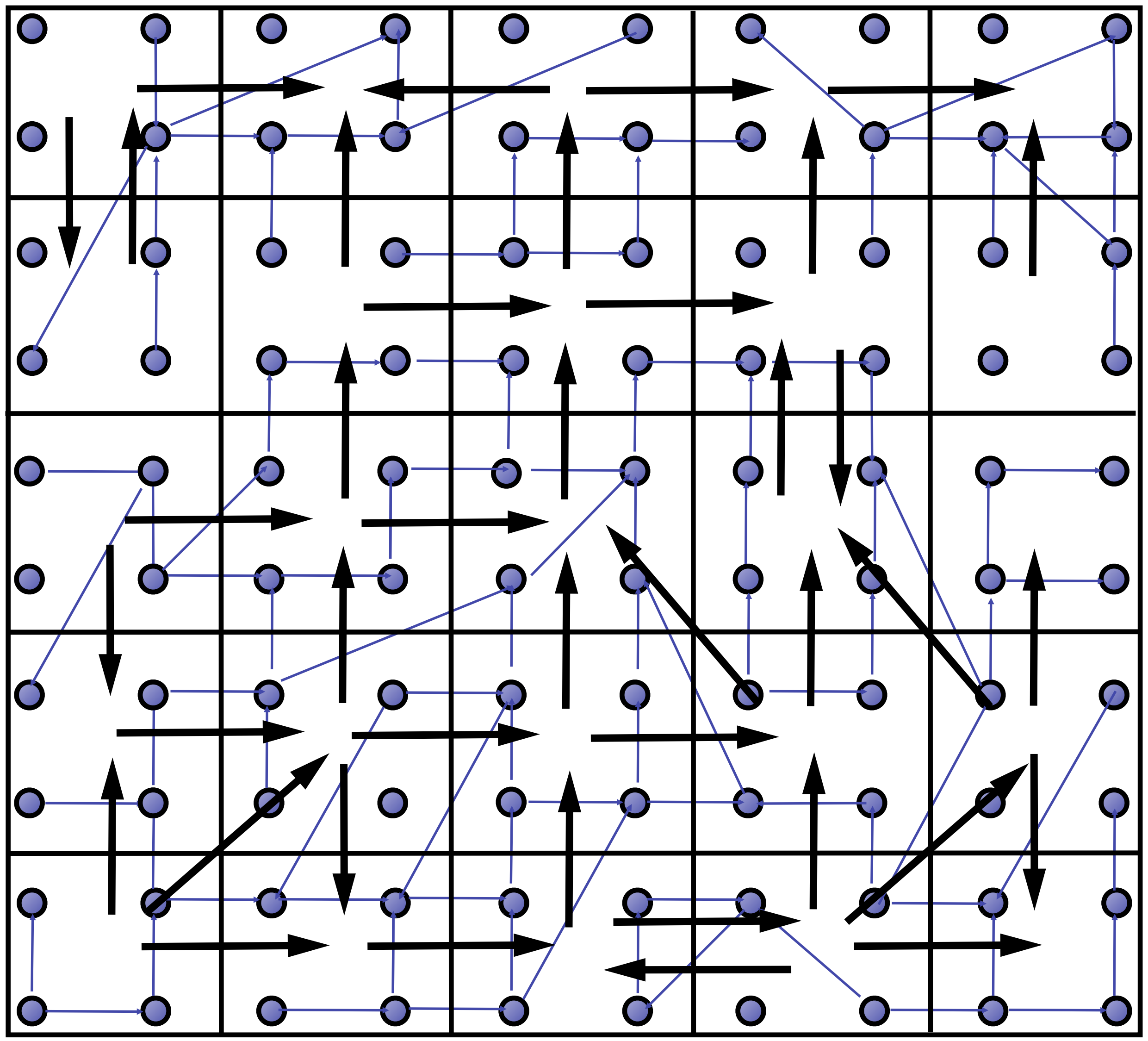

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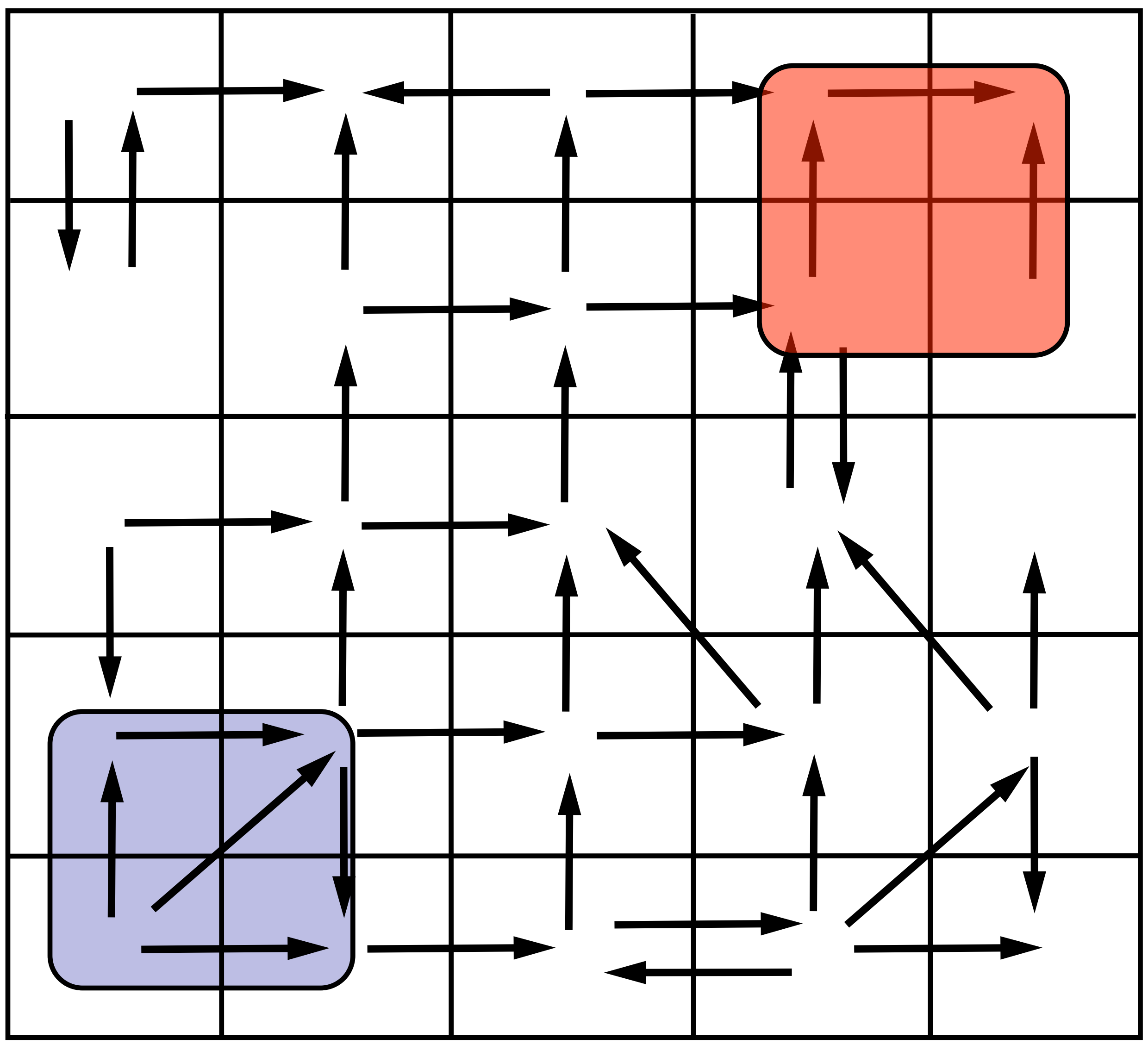

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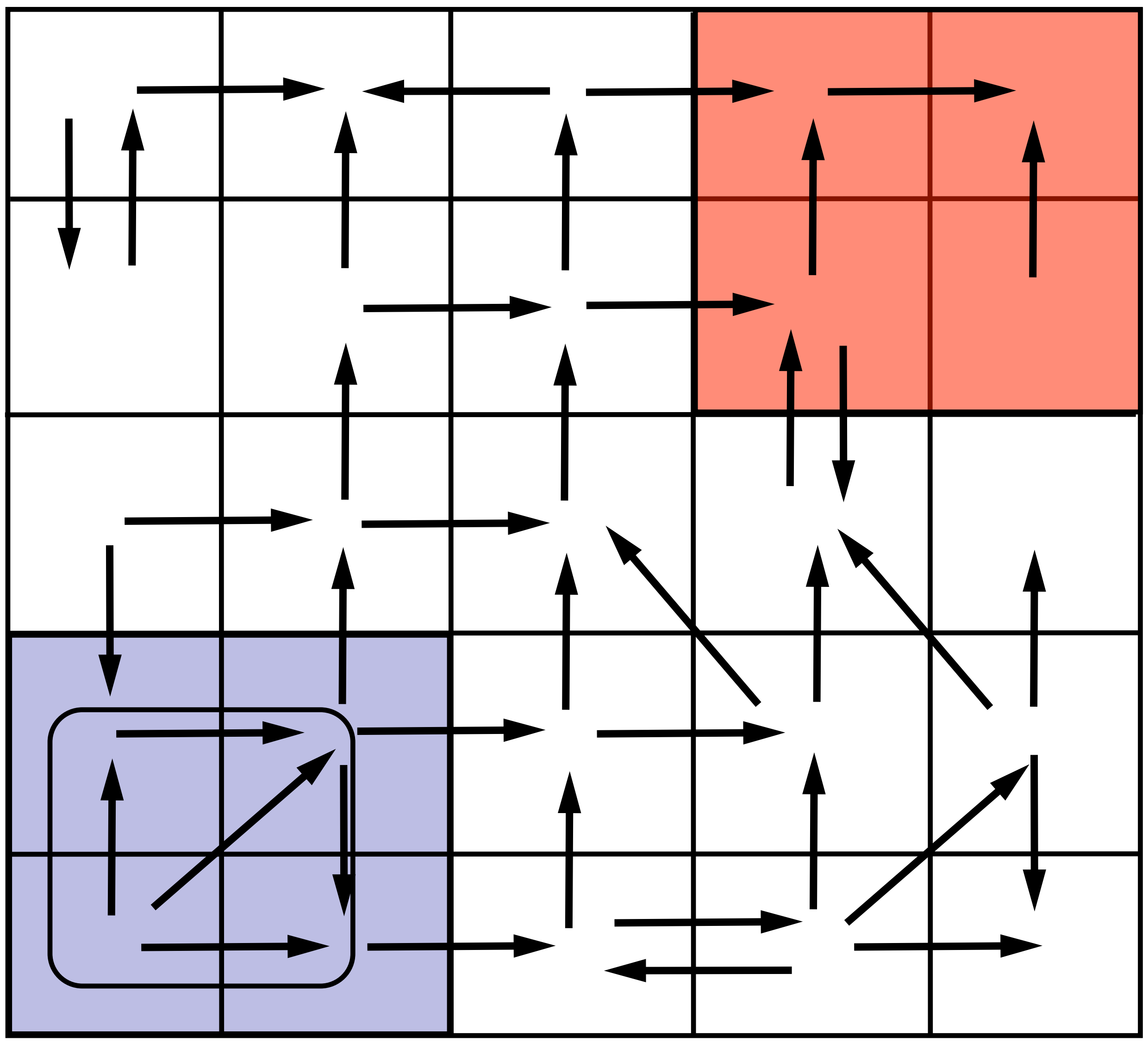

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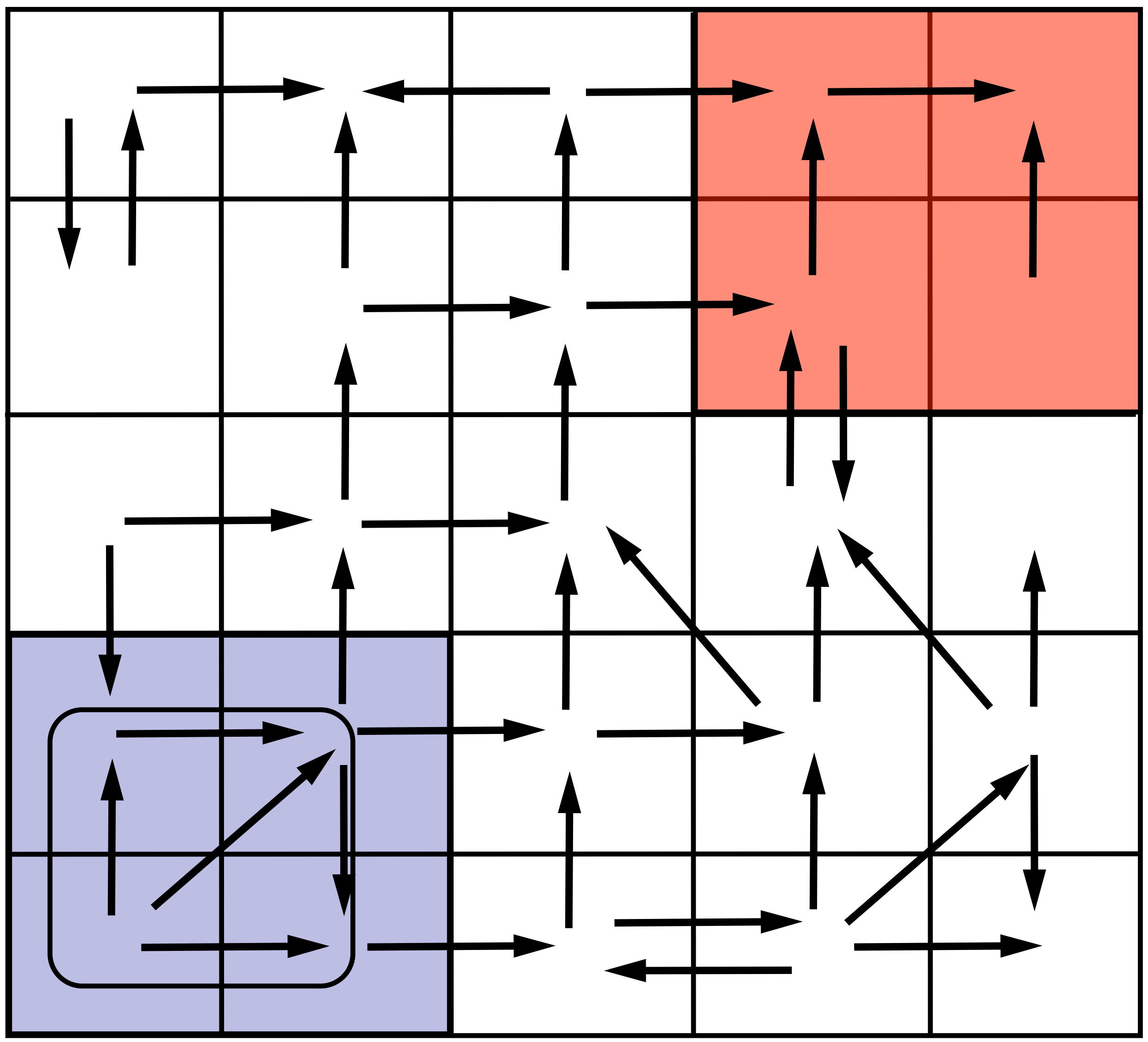
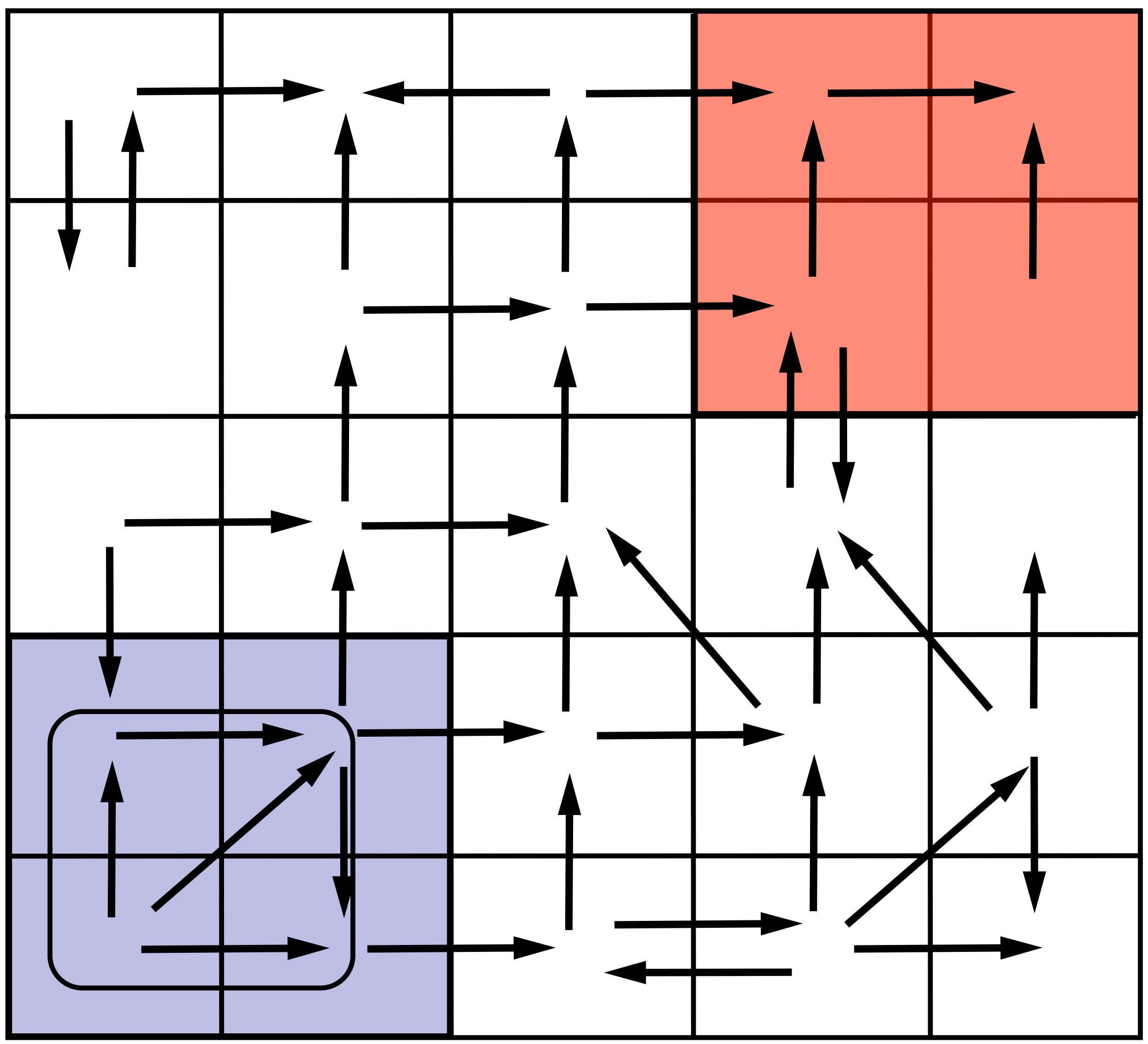


Analyze the abstract graph

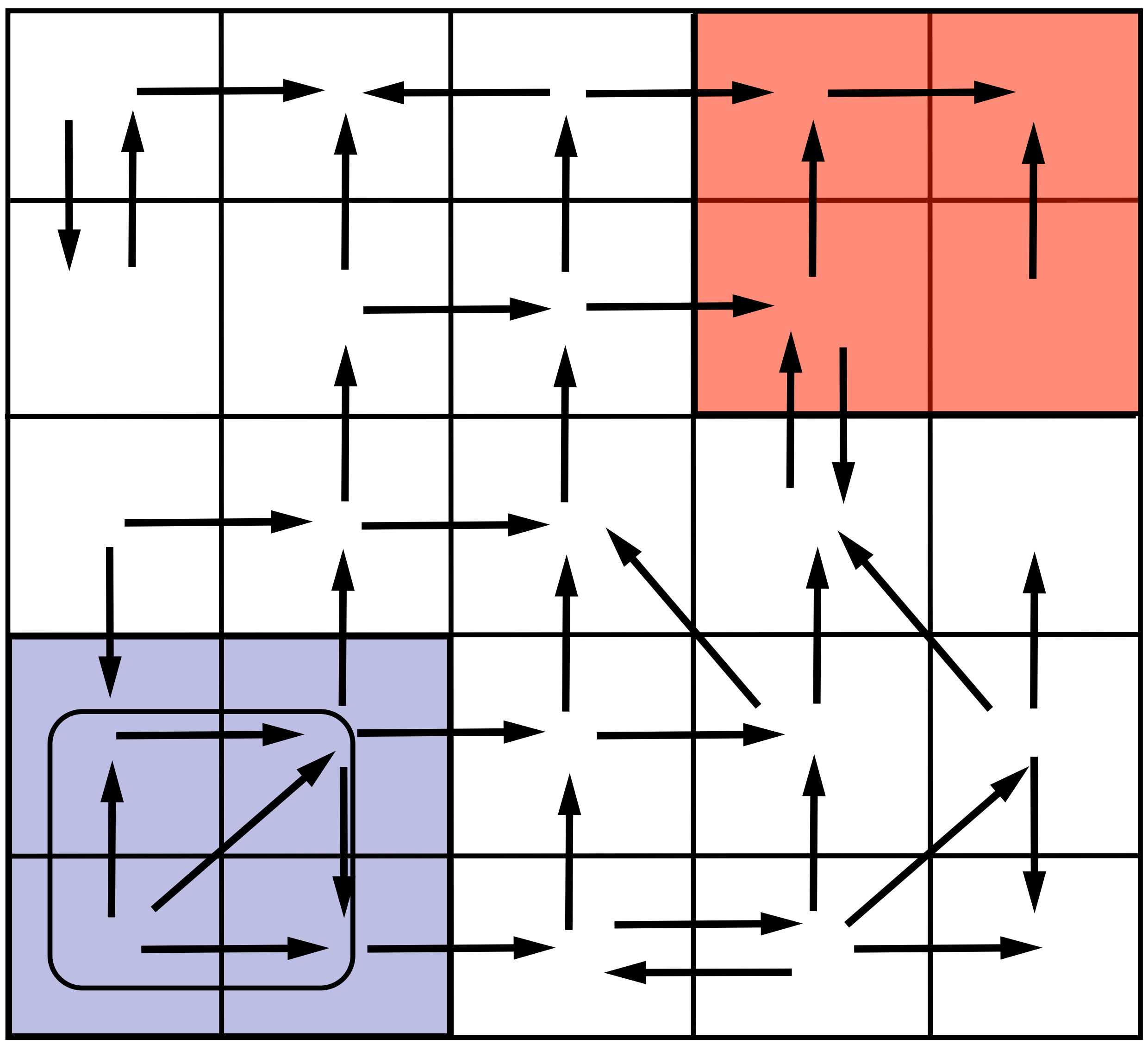
State-space partitioning



Analyze the abstract graph

Overapproximation:

State-space partitioning

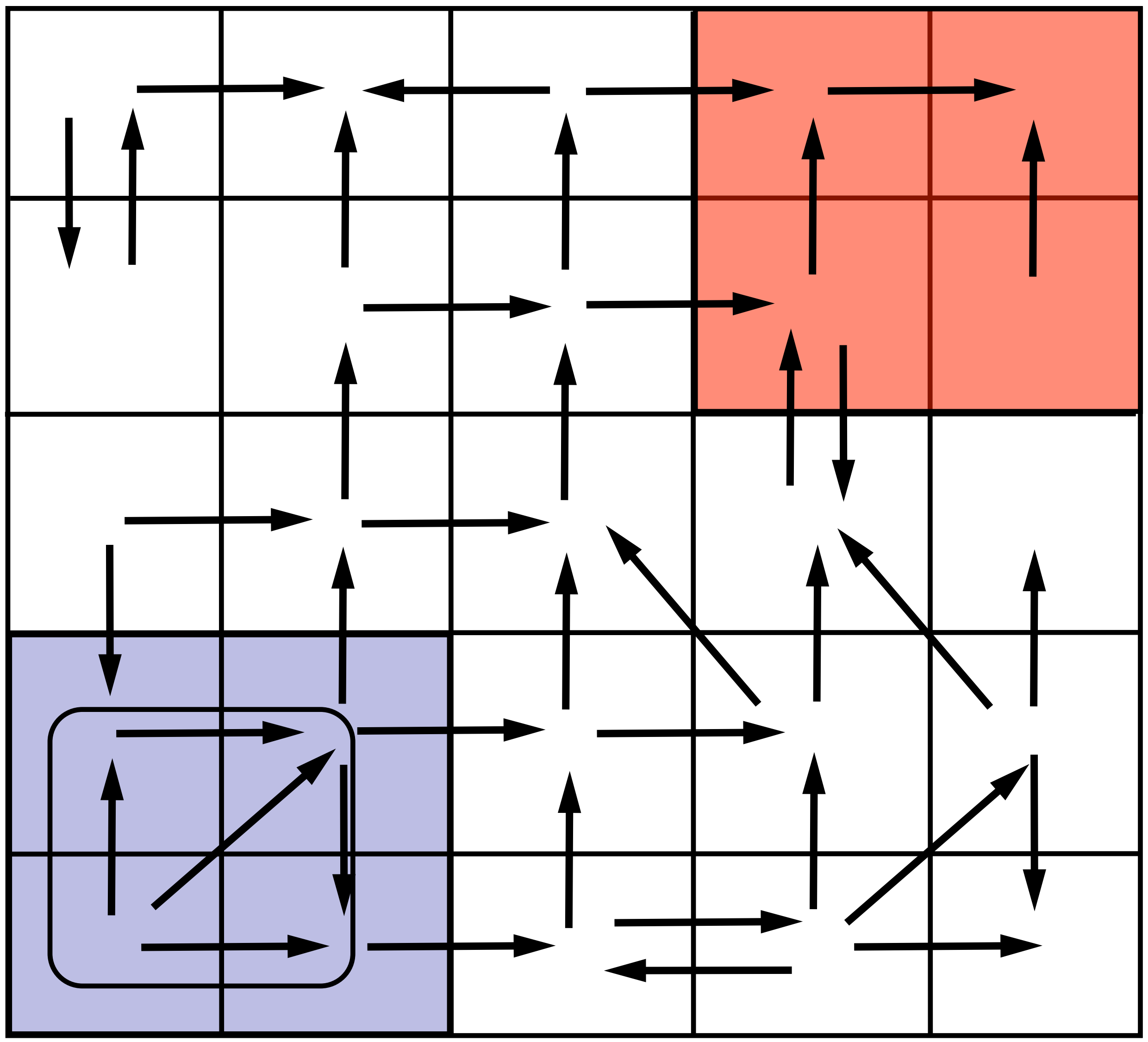


Analyze the abstract graph

Overapproximation:

Safe ⇒ System Safe

State-space partitioning



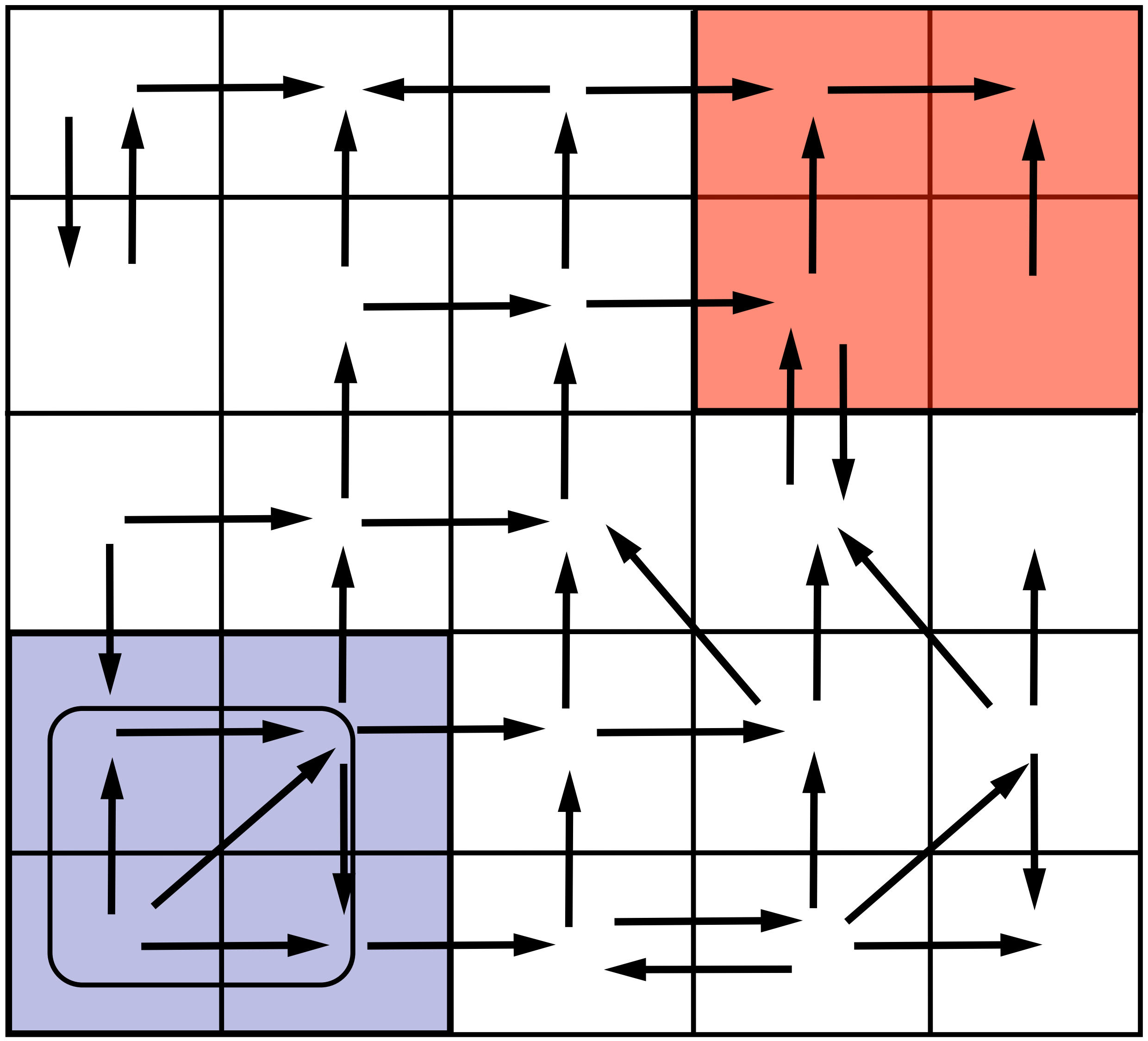
Analyze the abstract graph

Overapproximation:

Safe ⇒ System Safe

No false positives

State-space partitioning



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Overapproximation:

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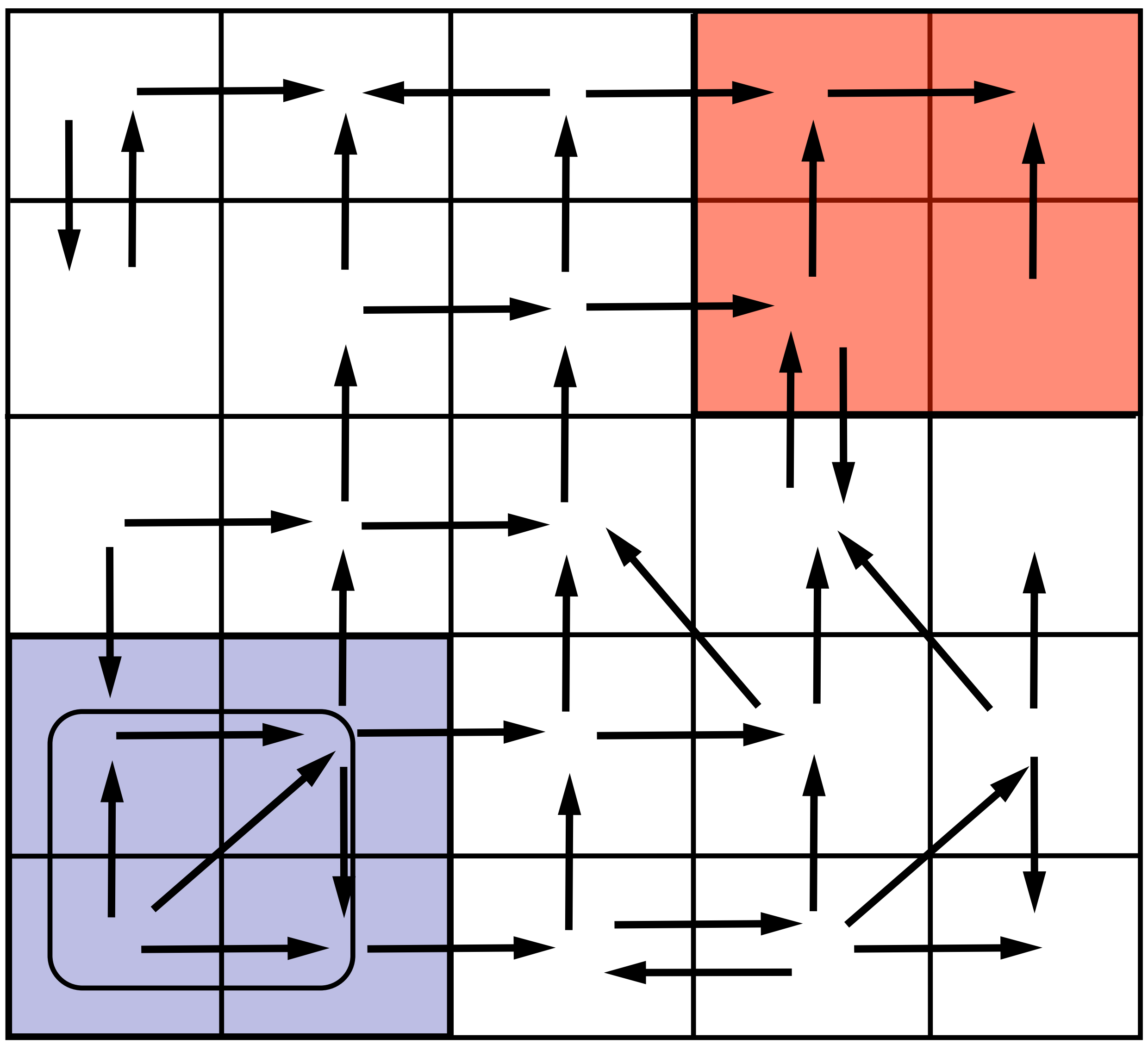
No false positives

Problem

State-space partitioning



Analyze the abstract graph

Overapproximation:

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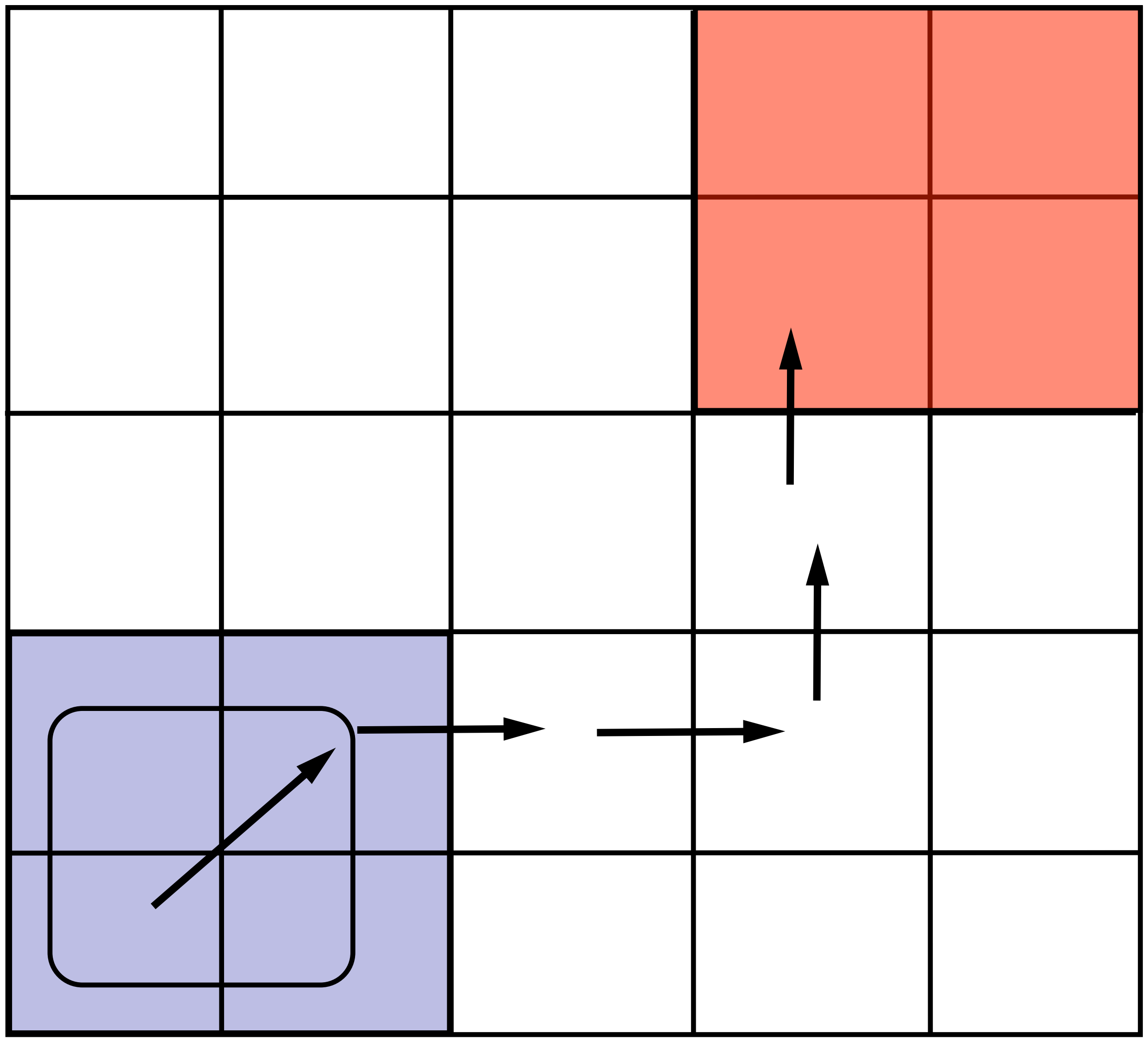
No false positives

Problem

Spurious counterexamples

State-space partitioning



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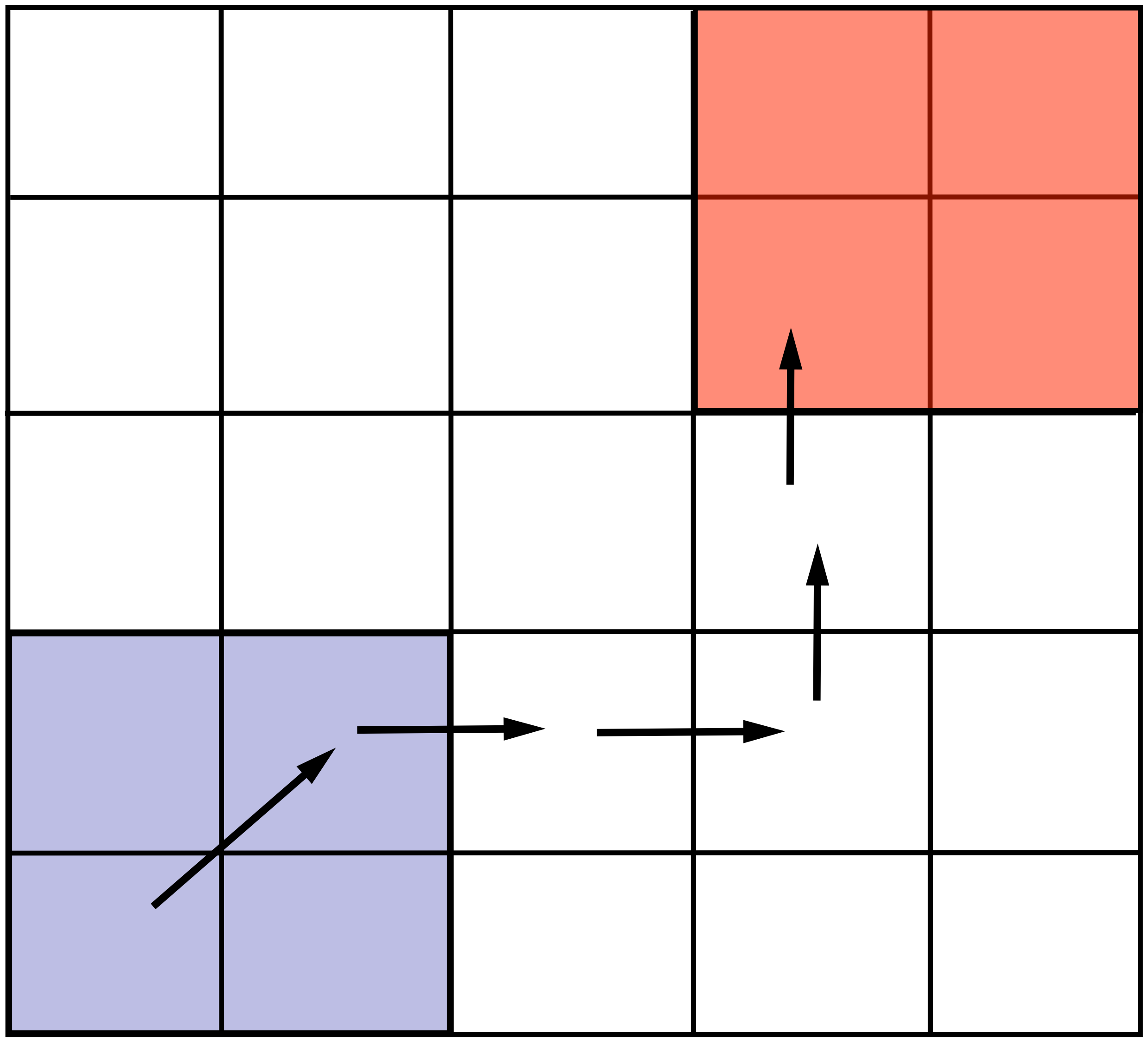
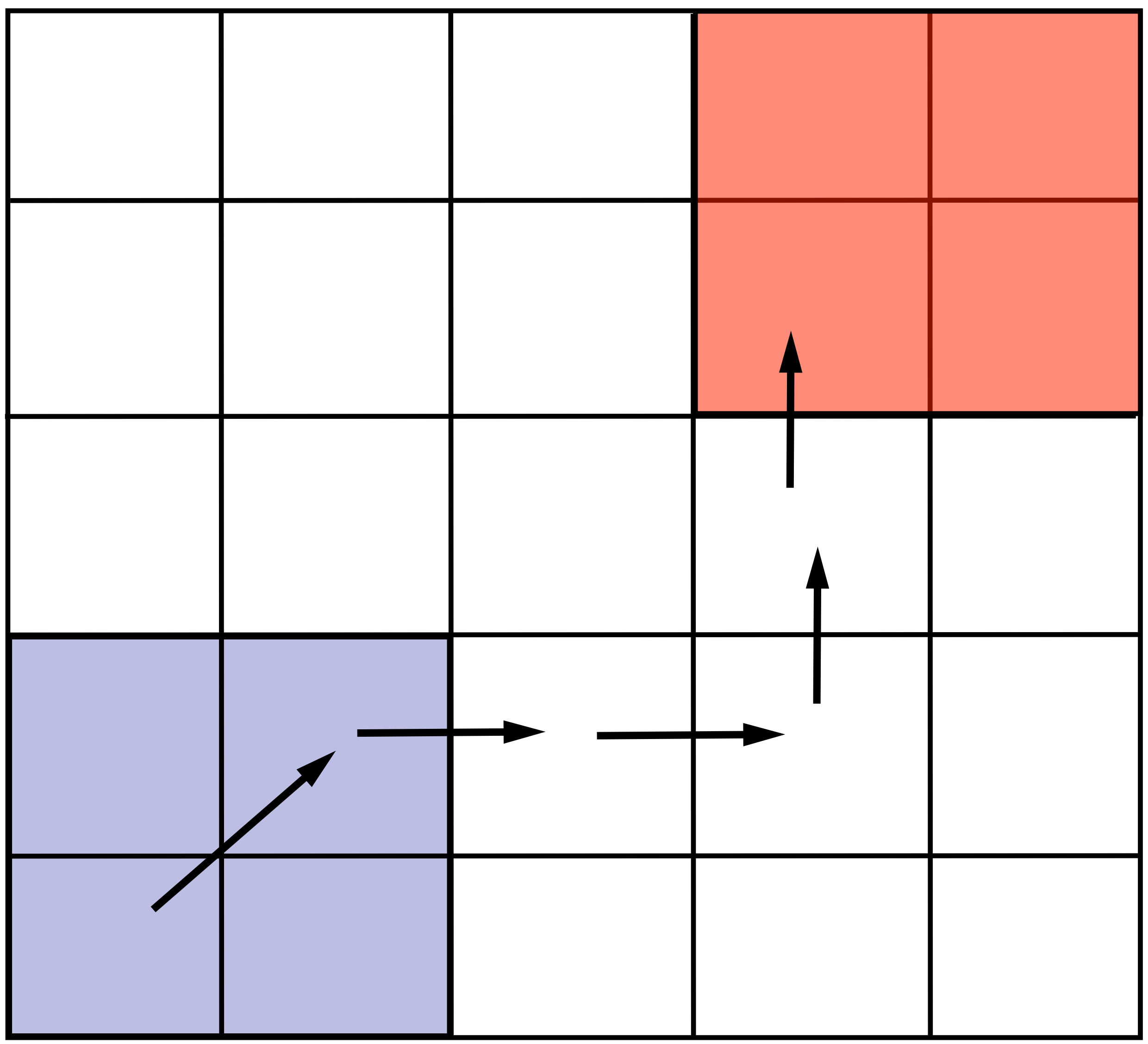


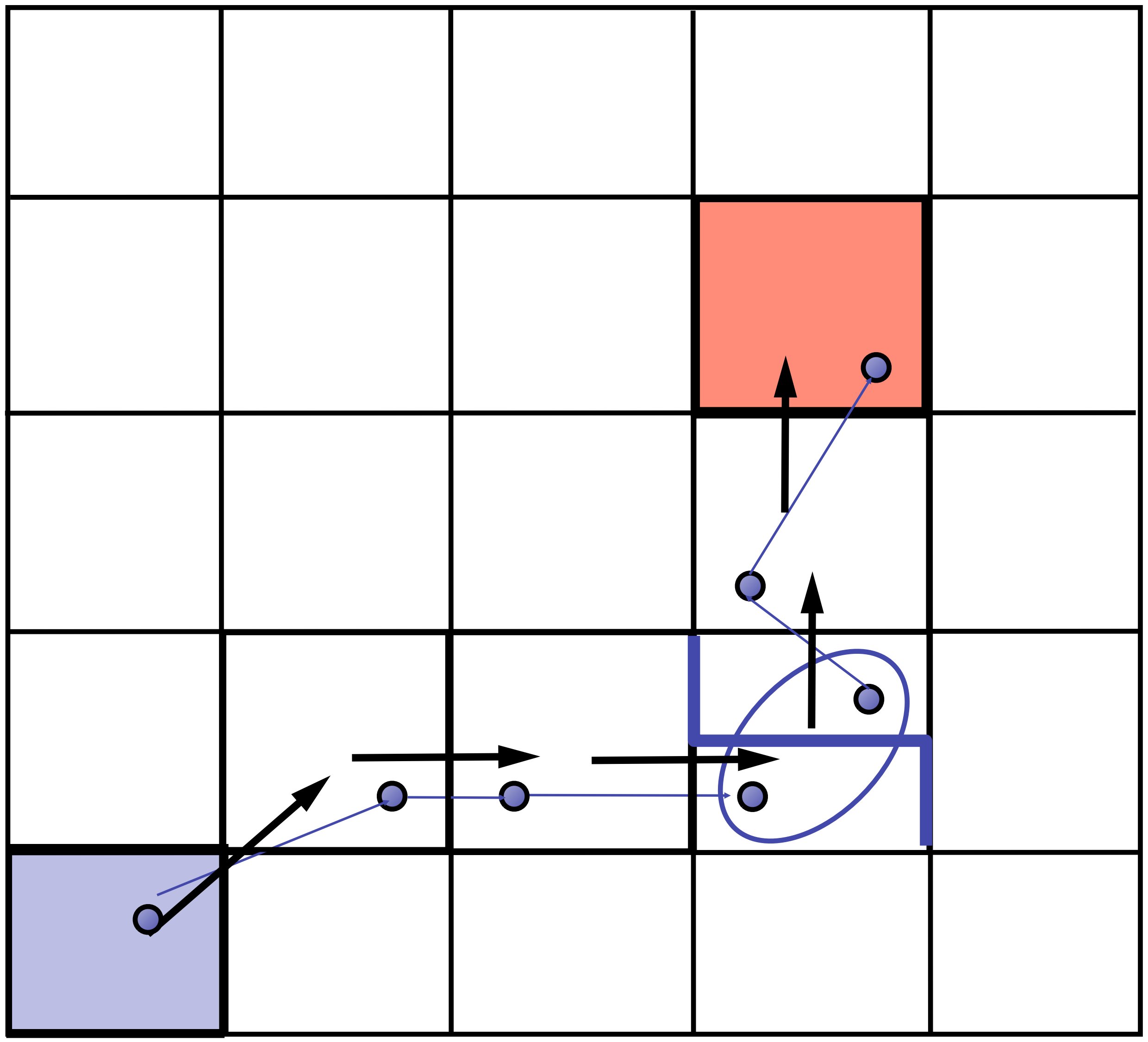
Counterex.-Guided Refinement

[Kurshan et al93] [Clarke et al 00][Ball-Rajamani 01]



Solution

Use spurious counterexamples to refine abstraction !



Solution

Use spurious **counterexamples** to **refine** abstraction

1. **Add predicates** to distinguish states across **cut**
2. Build **refined** abstraction

Imprecision due to **merge**

Counterex.-Guided Refinement

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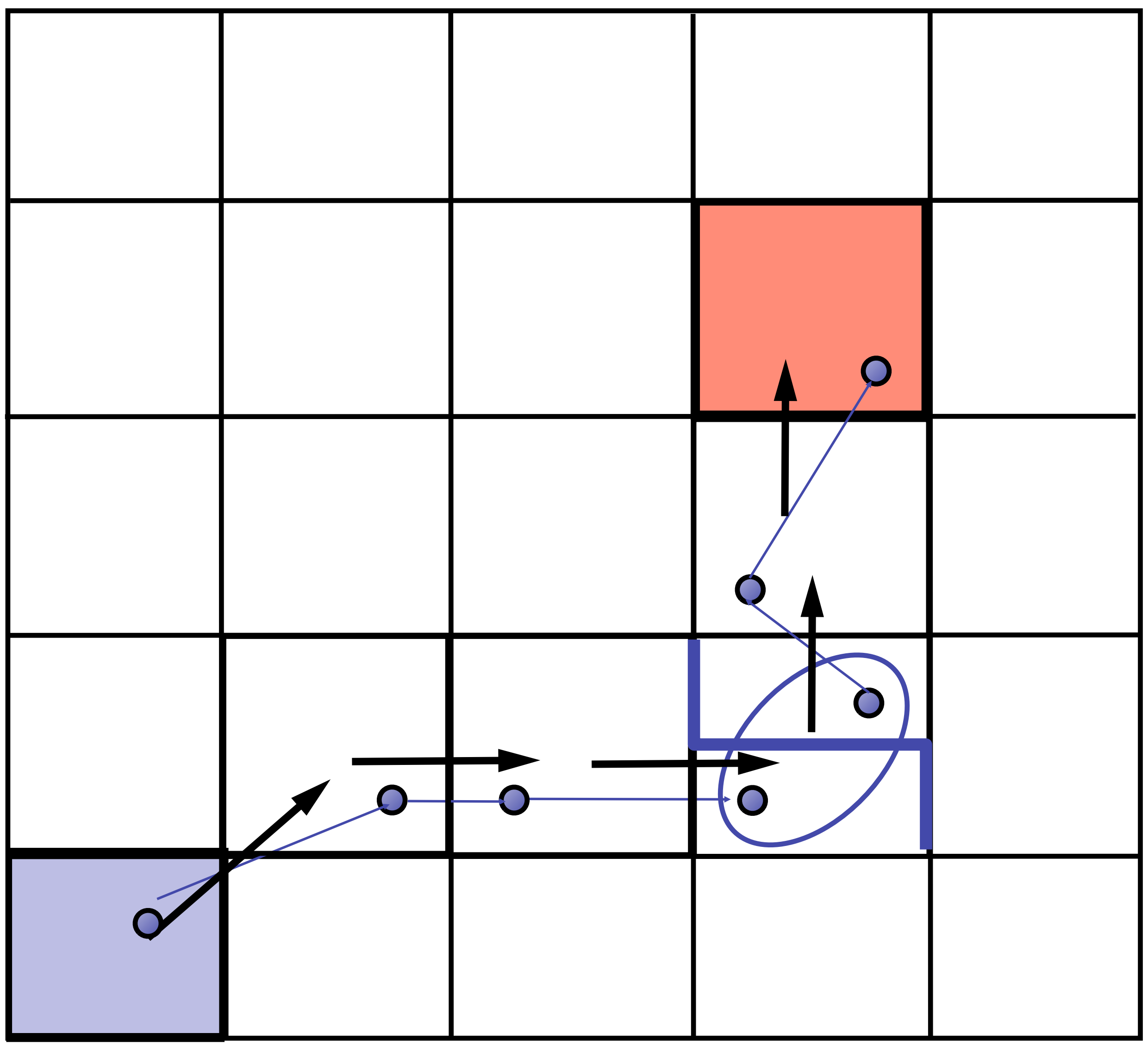

Solution

Use spurious **counterexamples** to **refine** abstraction

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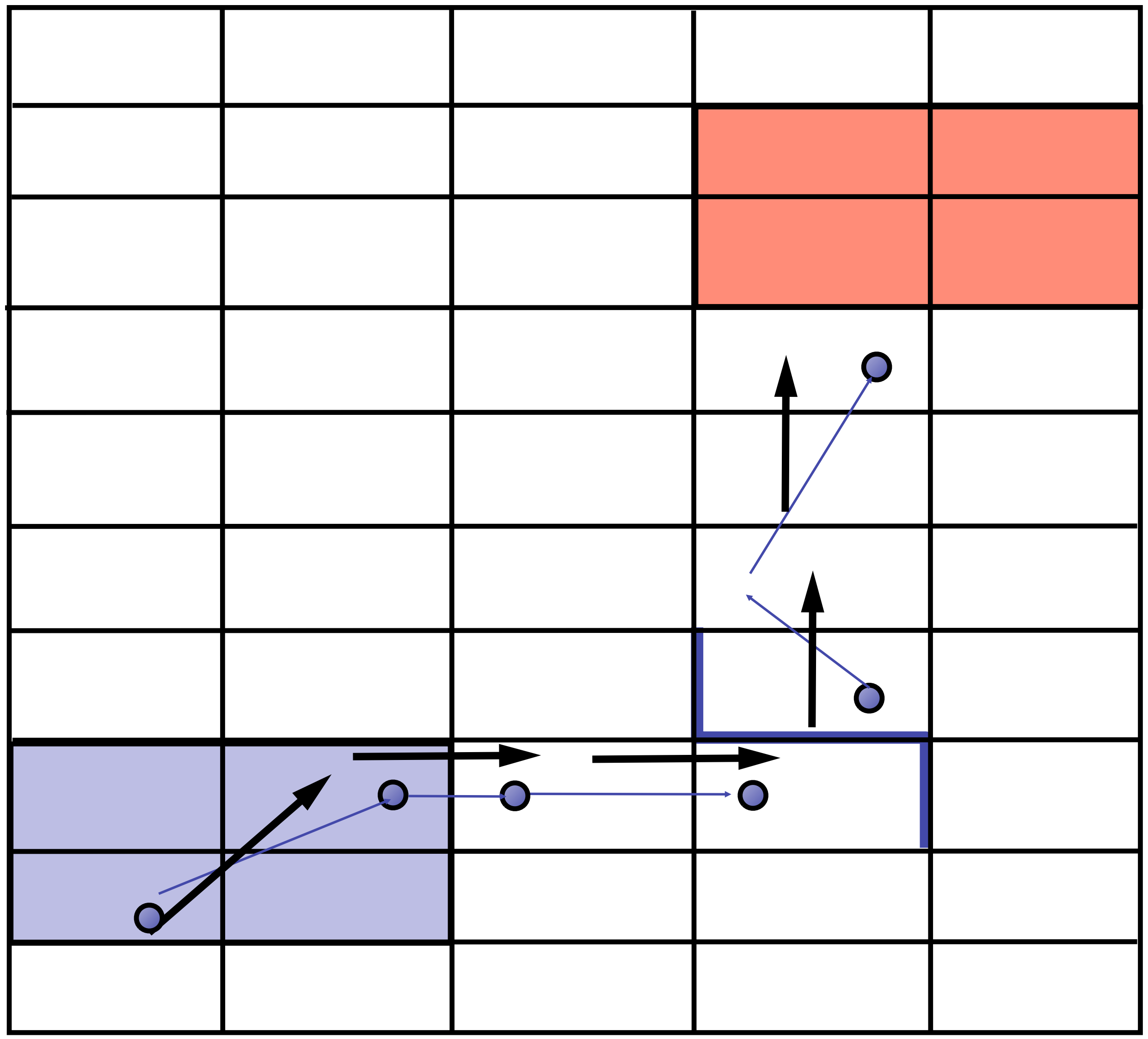
Iterative Abstraction-Refinement

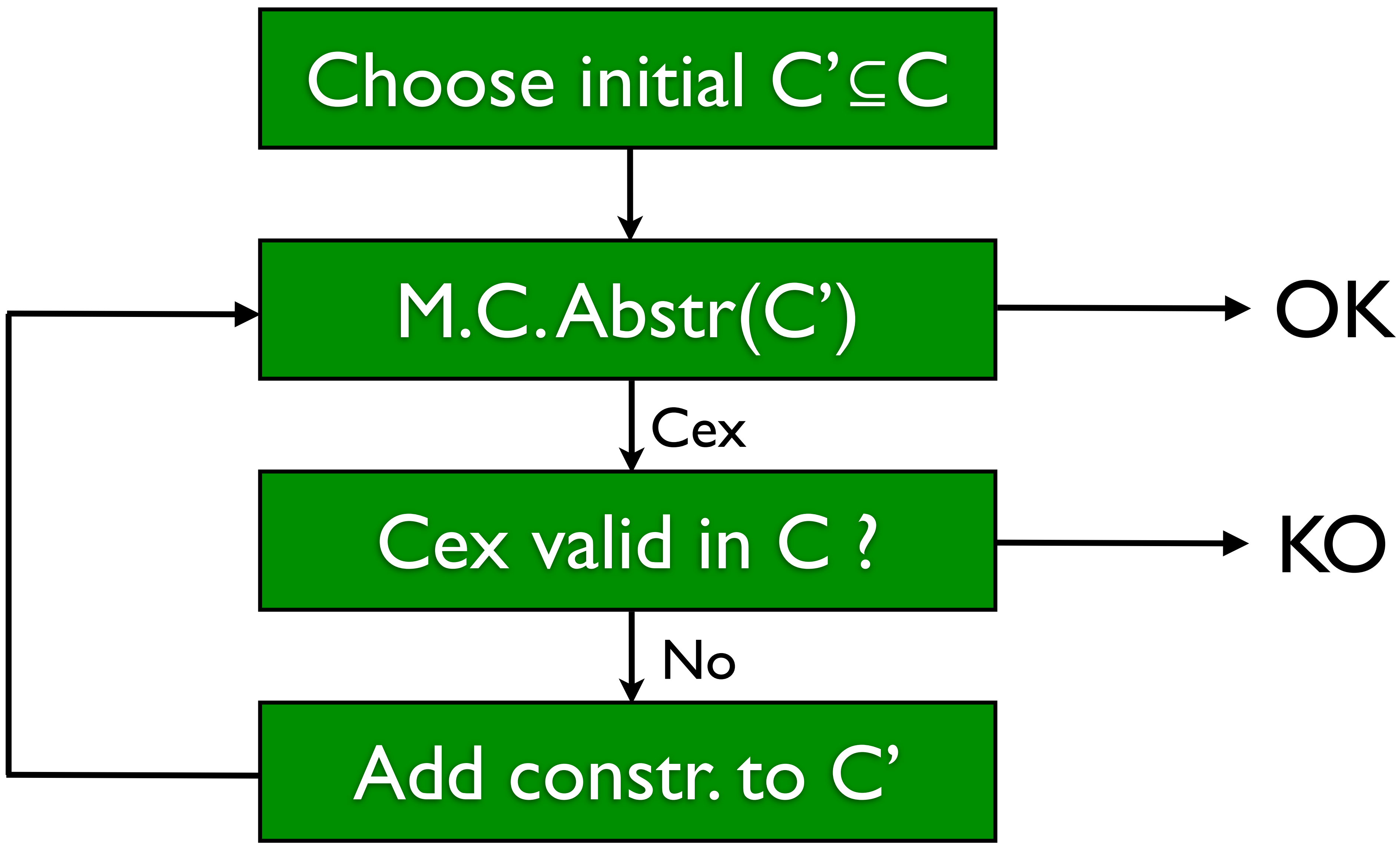


Solution

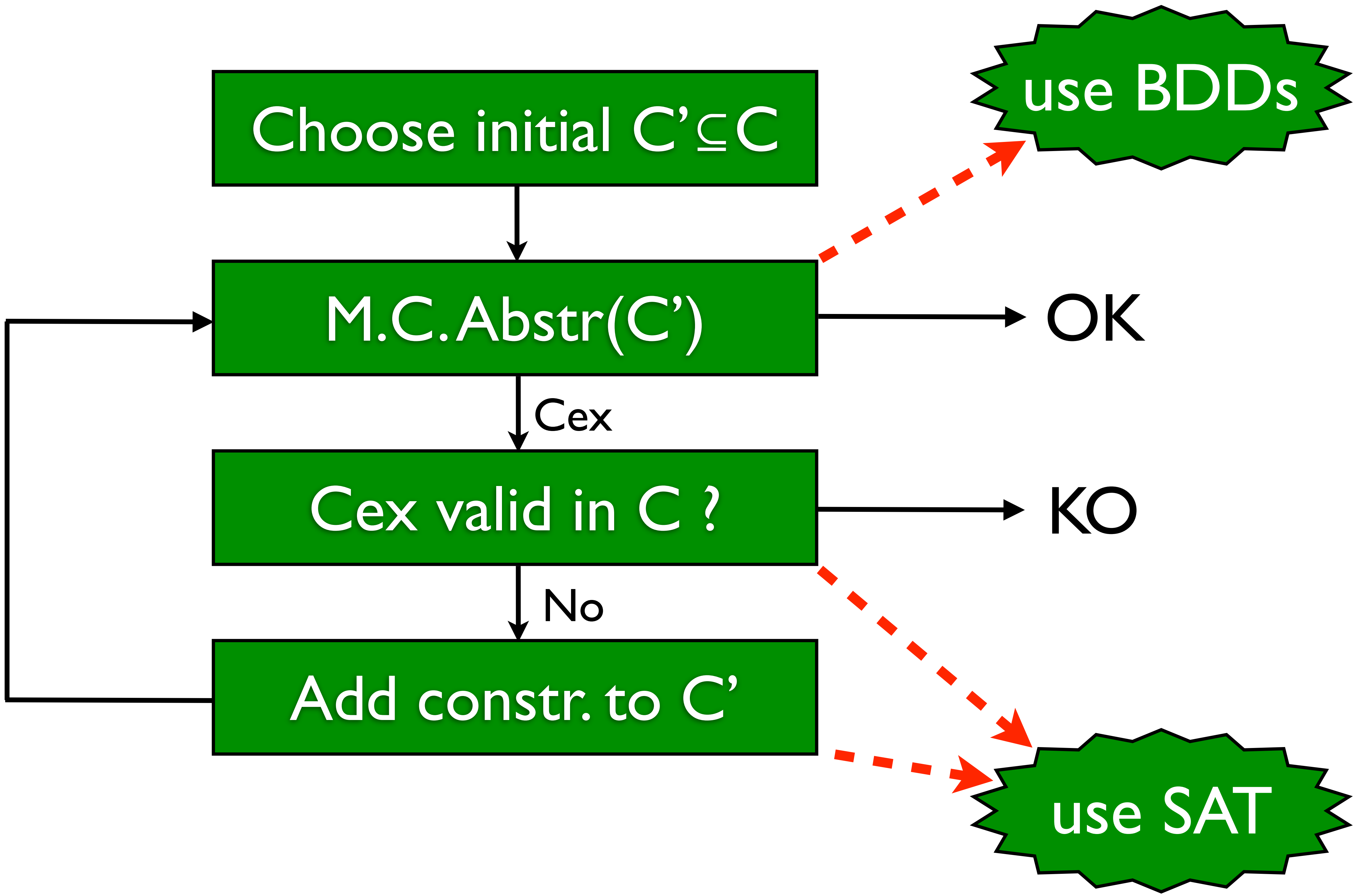
Use spurious **counterexamples** to **refine** abstraction

1. Add predicates to distinguish states across **cut**
2. Build **refined** abstraction
   -eliminates counterexample
3. **Repeat** search
   Till real counterexample or system proved safe

Abstraction refinement



Abstraction refinement



Abstract Cex - Safety

- **Abstract variables** Y=Support(C',I,F)
- Abstract system is model-checked using BDD-based symbolic MC with variables in Y only and $|Y| \ll |X|$
- Abstract counter-example is a truth assignment to $\{ x_t \mid x \in Y \wedge 0 \leq t \leq \mathbf{k} \}$ where k is the number of steps in the counter-example

Concretization of Cex

- The abstract Cex $\mathbf{A}^{\alpha}$ satisfies:

  $I(Y_0) \wedge T_{0..k-1}(Y_0,\ldots,Y_{k-1}) \wedge \bigvee_{i=0..k-1} \mathbf{Bad}(Y_i)$

- Search for a concrete A consistent with $\mathbf{A}^{\alpha}$:

  $\mathbf{A}^{\alpha}\mathbf{(Y)} \wedge I(X_0) \wedge T_{0..k-1}(X_0,\ldots,X_{k-1}) \wedge \bigvee_{i=0..k-1} \mathbf{Bad}(X_i)$

  =BMC but guided by the abstract Cex

- If unsat Cex cannot be made concrete and it is thus spurious

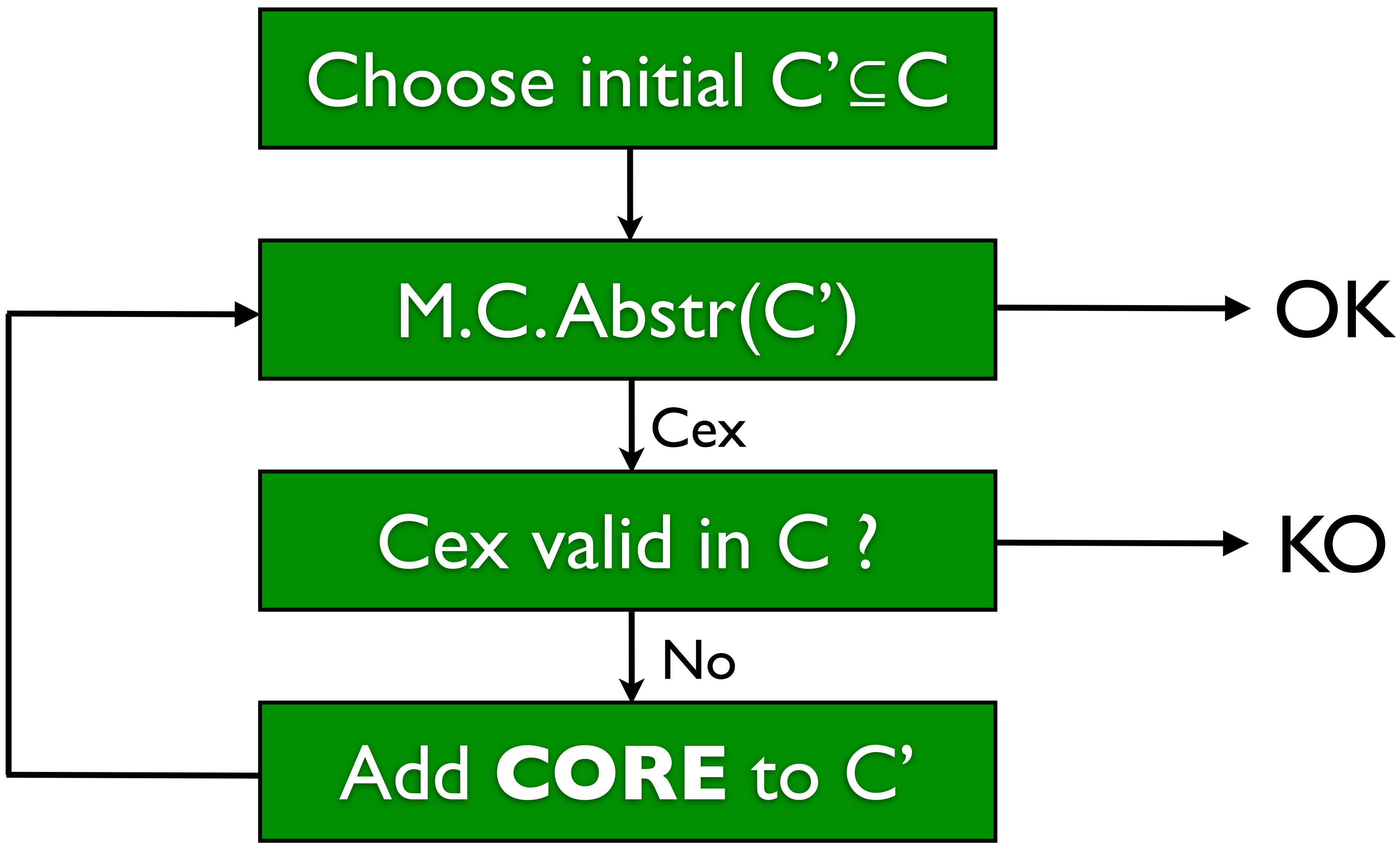
Refinement

- Refinement: add constraints to C'
- Goal: rule out the Cex in the next abstract model
- There are many technics for that
- One based on SAT machinery: use **resolution based refutation** of the unsat formula underlying the concretization of the abstract counter-example

Resolution based refinement

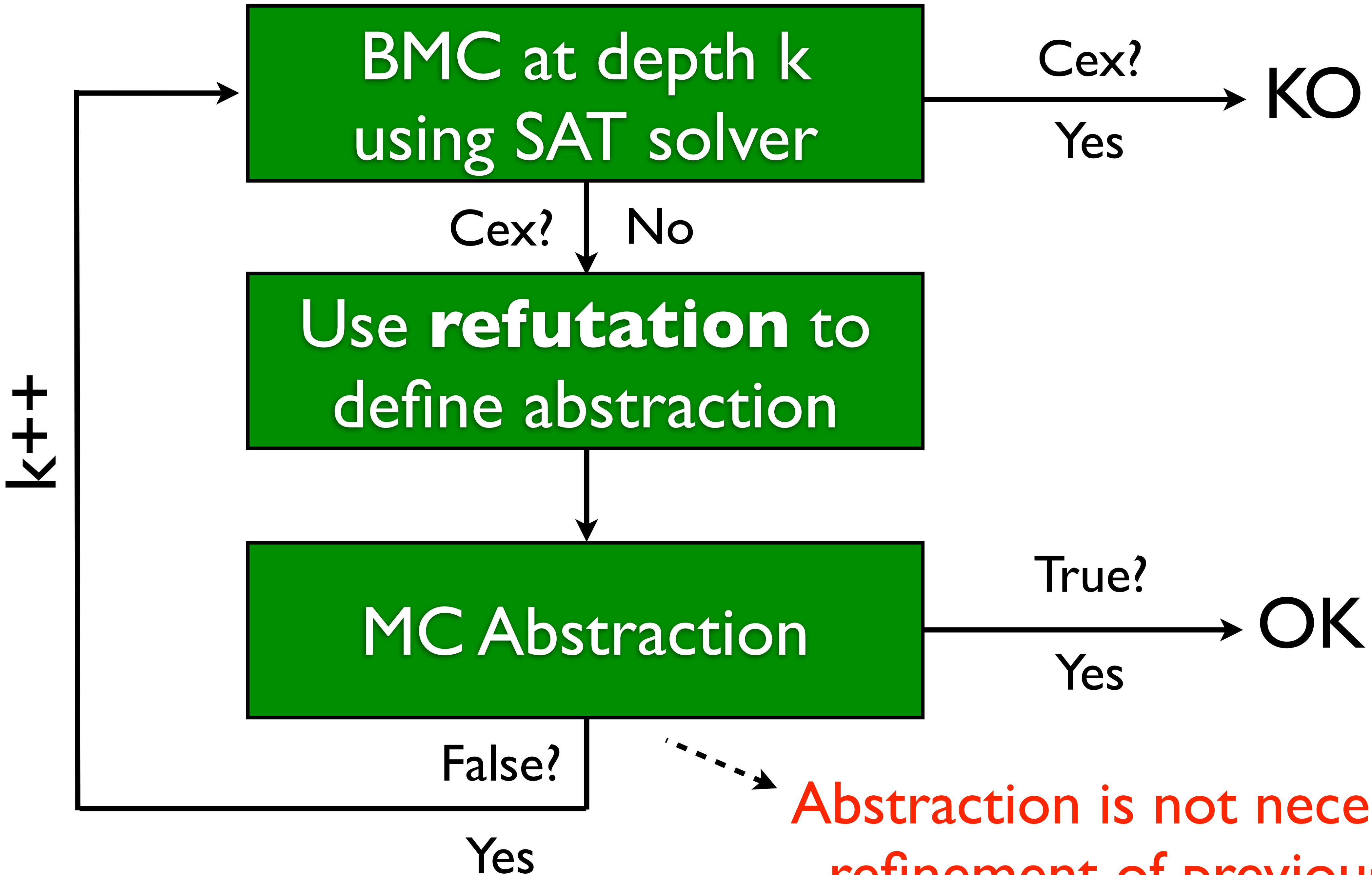
- $A^{\alpha}(Y) \wedge I(X_0) \wedge T_{0..k-1}(X_0,\ldots,X_{k-1}) \wedge \vee_{i=0..k-1} Bad(X_i)$ is **unsatisfiable**
- SAT solver returns unsatisfiable and produce an **UNSAT core** CORE
- A^{α} cannot be extended to a concrete Cex: CORE is sufficient to prove it
- Add CORE to C’

Abstraction refinement



Variation [McMillan03]

Conclude when k is large enough



Abstraction is not necessary a refinement of previous one

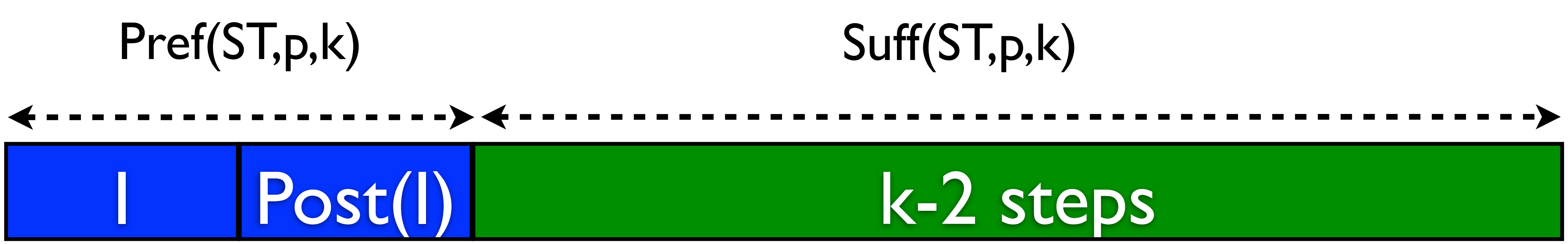
Interpolation based unbounded Sat-based model-checking [McMillan03]

Interpolant

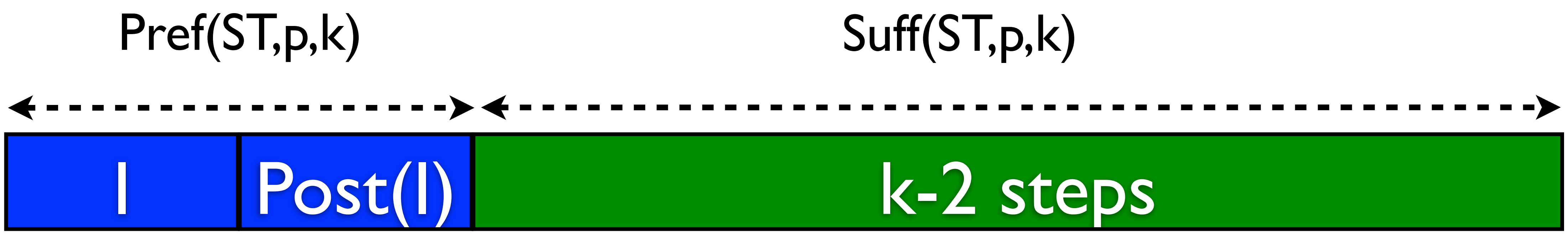
- An **interpolant** $\mathbb{I}$ for an unsatisfiable formula $A \wedge B$ is a formula such that
  - $A \Longrightarrow \mathbb{I}$
  - $\mathbb{I} \wedge B$ is unsatisfiable
  - $\mathbb{I}$ only refers to the common variables of A and B
- Ex: $A \equiv p \wedge q$, $B \equiv \neg q \wedge r$, $\mathbb{I} \equiv q$

Interpolation and SAT-MC

- First, call **BMC**(ST,p,k)
- Decompose BMC(ST,p,k) into Pref(ST,p,k)$\wedge$Suff(ST,p,k), where
  - Pref(ST,p,k)$\equiv$init+first transition
  - Suff(ST,p,k)$\equiv$k-1 last transitions+$\neg$p
  - if formula is SAT, we have Cex
- Otherwise, compute $\mathbb{I}$ for Pref(ST,p,k)$\wedge$Suff(ST,p,k



Interpolation and SAT-MC



Fact: the interpolant $\mathbb{I}$ **overapproximates** the set of initial states and those accessible in one step and that do **not** lead to bad states within k steps (quality of the overapproximation)

Idea: iterate from a new set of initial states : $\mathbb{I}$

Interpolation procedure

procedure interpolation (M, p)
1. initialize k
2. while *true* do
3.   if $BMC(M, p, k)$ is SAT then return *counterexample*
4. $R = I$
5.   while true do
6. $M' = (S, R, T, L)$
7.   let $C = Pref(M', p, k) \wedge Suff(M', p, k)$
8.   if C is SAT then break (goto line 15)
9.   /* C is UNSAT */
10.   compute interpolant $\mathcal{I}$ of $Pref(M', p, k) \wedge Suff(M', p, k)$
11. $R' = \mathcal{I}$ is an over-approximation of states reachable from R in one step.
12.   if $R \Rightarrow R'$ then return *verified*
13. $R = R \vee R'$
14.   end while
15.   increase k
16. end while
end

Interpolation procedure

Discover negative instances

procedure interpolation (M, p)
1. initialize k
2. while *true* do
3. if $BMC(M, p, k)$ is SAT then return *counterexample*
4. $R = I$
5. while true do
6. $M' = (S, R, T, L)$
7. let $C = Pref(M', p, k) \wedge Suff(M', p, k)$
8. if C is SAT then break (goto line 15)
9. /* C is UNSAT */
10. compute interpolant $\mathcal{I}$ of $Pref(M', p, k) \wedge Suff(M', p, k)$
11. $R' = \mathcal{I}$ is an over-approximation of states reachable from R in one step.
12. if $R \Rightarrow R'$ then return *verified*
13. $R = R \vee R'$
14. end while
15. increase k
16. end while

end

Interpolation procedure

procedure interpolation (M, p)
1. initialize k
2. while *true* do
3. if $BMC(M, p, k)$ is SAT then return *counterexample*
4. $R = I$
5. while true do
6. $M' = (S, R, T, L)$
7. let $C = \mathit{Pref}(M', p, k) \wedge \mathit{Suff}(M', p, k)$
8. if C is SAT then break (goto line 15)
9. /* C is UNSAT */
10. compute interpolant $\mathcal{I}$ of $\mathit{Pref}(M', p, k) \wedge \mathit{Suff}(M', p, k)$
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end

Potentially spurious counter-example due to over-approximation

Interpolation procedure

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Abstract fixpoint computation through interpolants

Interpolation procedure

procedure interpolation (M, p)
1. initialize k
2. while *true* do
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12. if $R \Rightarrow R'$ then return *verified*
13. $R = R \vee R'$
14. end while
15. increase k
16. end while

end

when k=diameter, the abstract algorithm concludes !

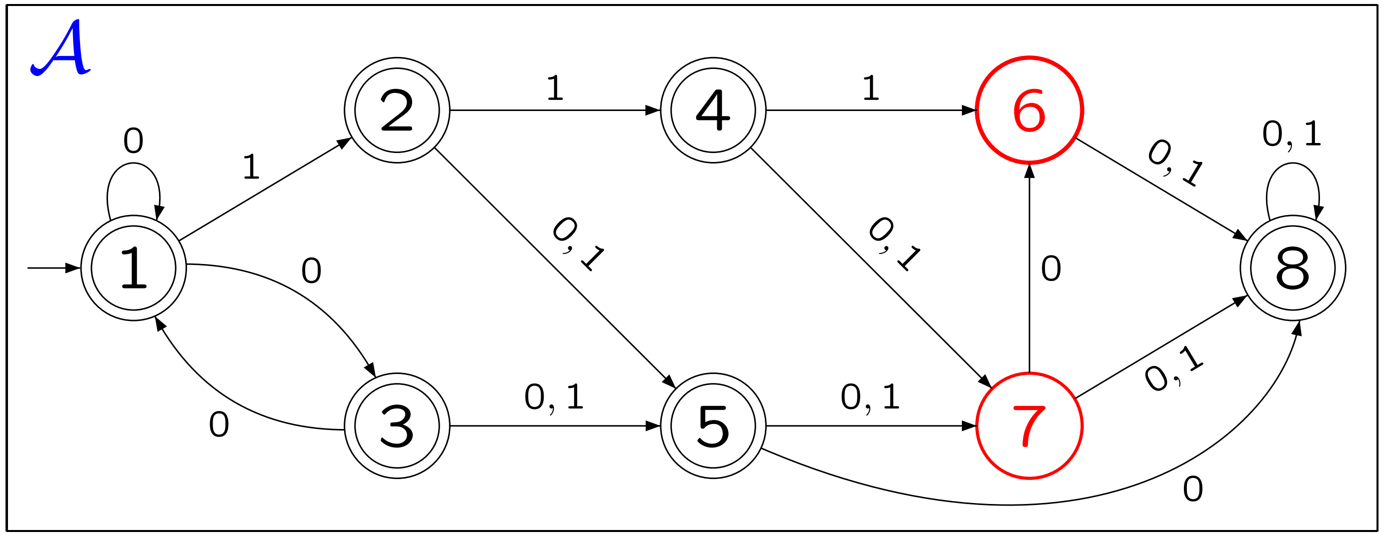
But most often it concludes **much earlier** !

This is a complete framework !

Discovering inductive invariants in subset constructions

Universality of NFA

- Nond. finite automata A=(Q,Σ,q0,δ,F)



- L(A)≠Σ* iff there exists a word w such that all runs on w end up in Q\F.
- Special case for L(A)⊆?L(B), PSpace-C.

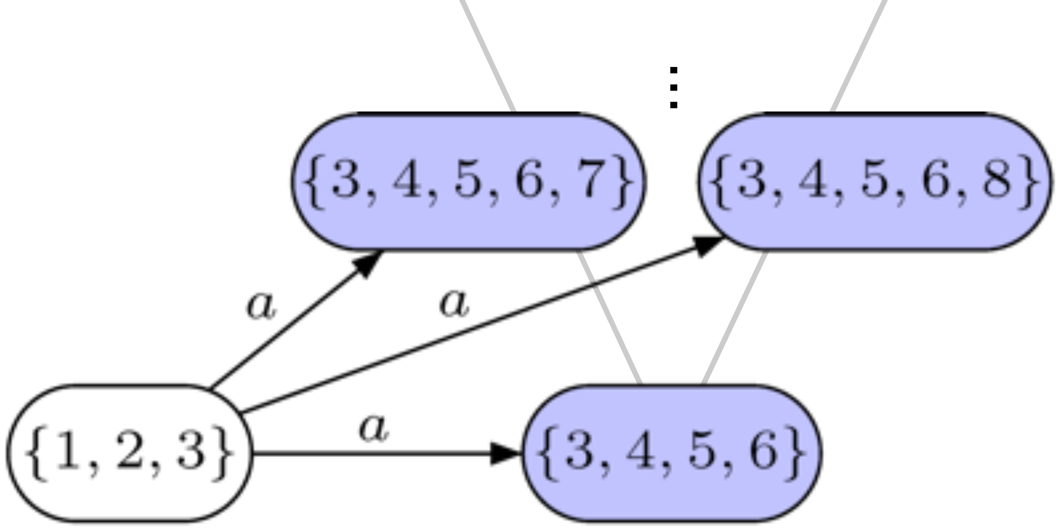
Universality of NFA

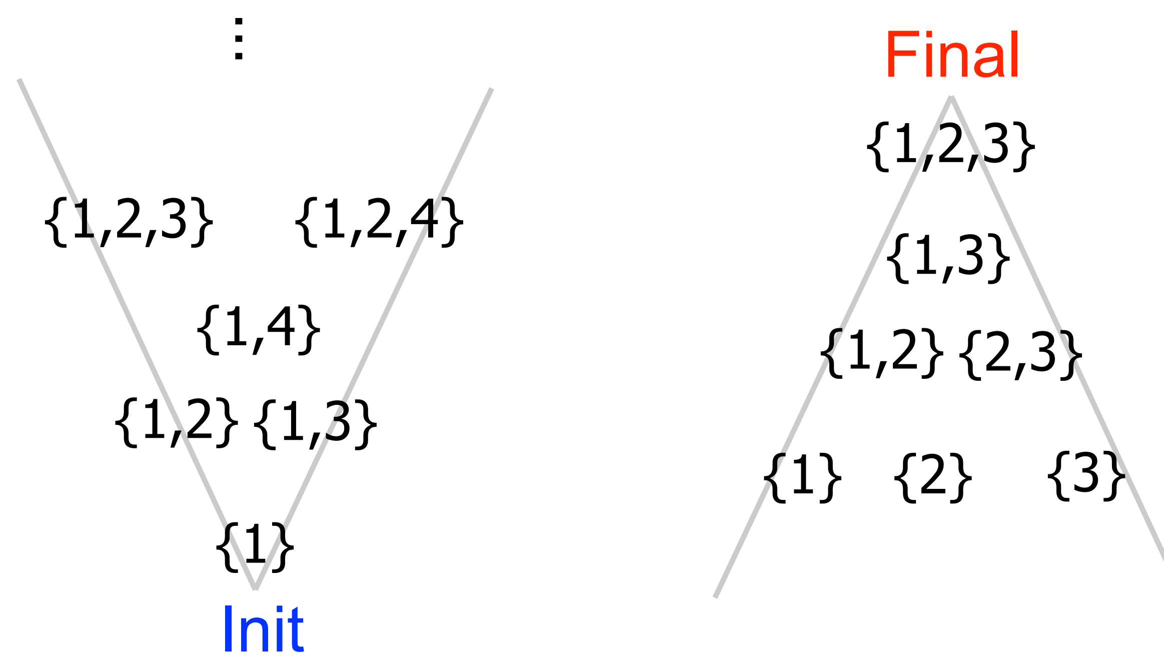
- Can be solved through reachability in STS (subset construction)
- **Hard** because one Boolean variable per state of the automaton - BDDs do not scale
- But special class of STS: monotonicity
- There are practical alternative algorithms to BDDs, based on antichains for example

“Closed” subset construction

Transition relation can be “closed” without changing the language.

A^c:

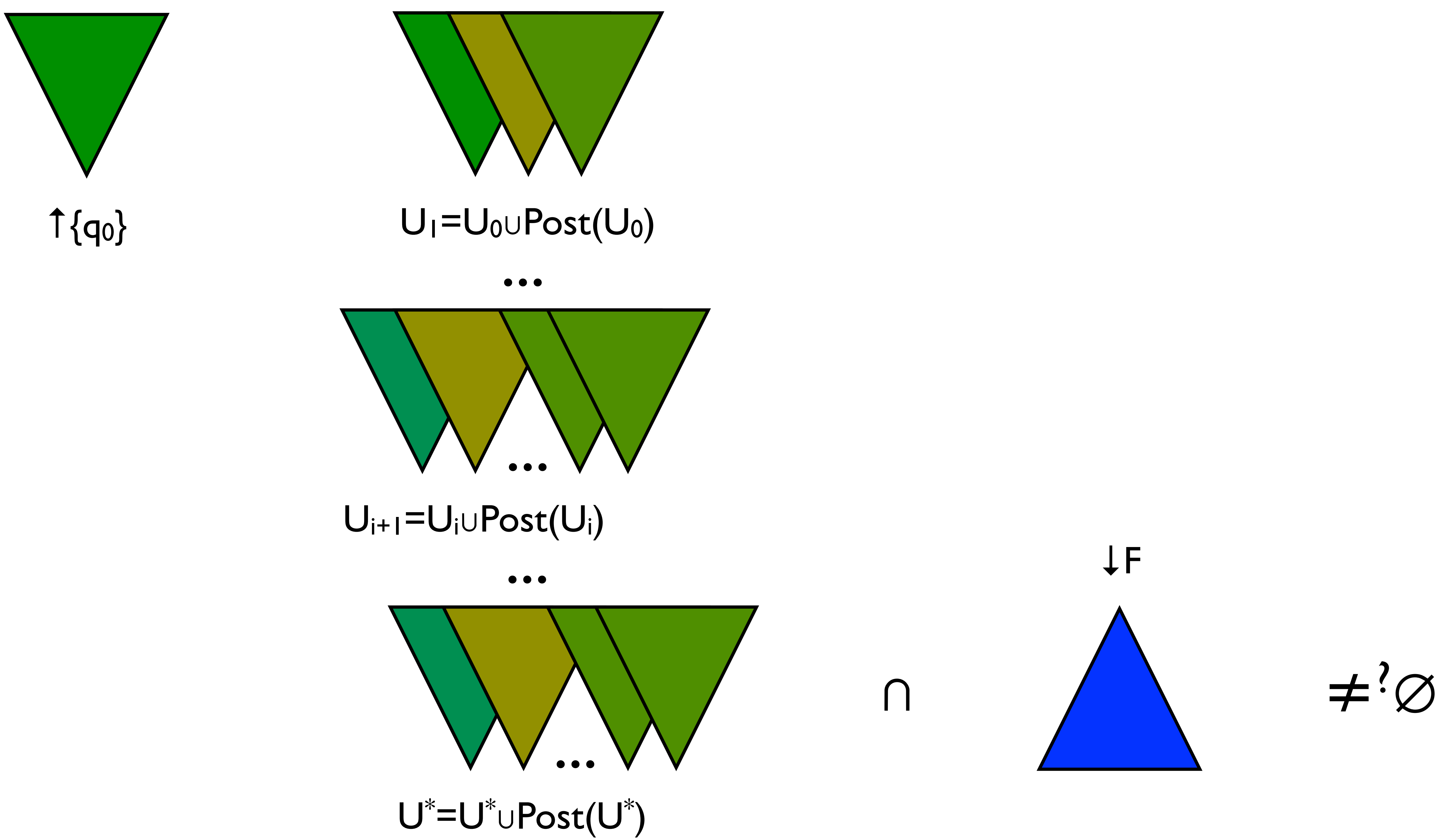




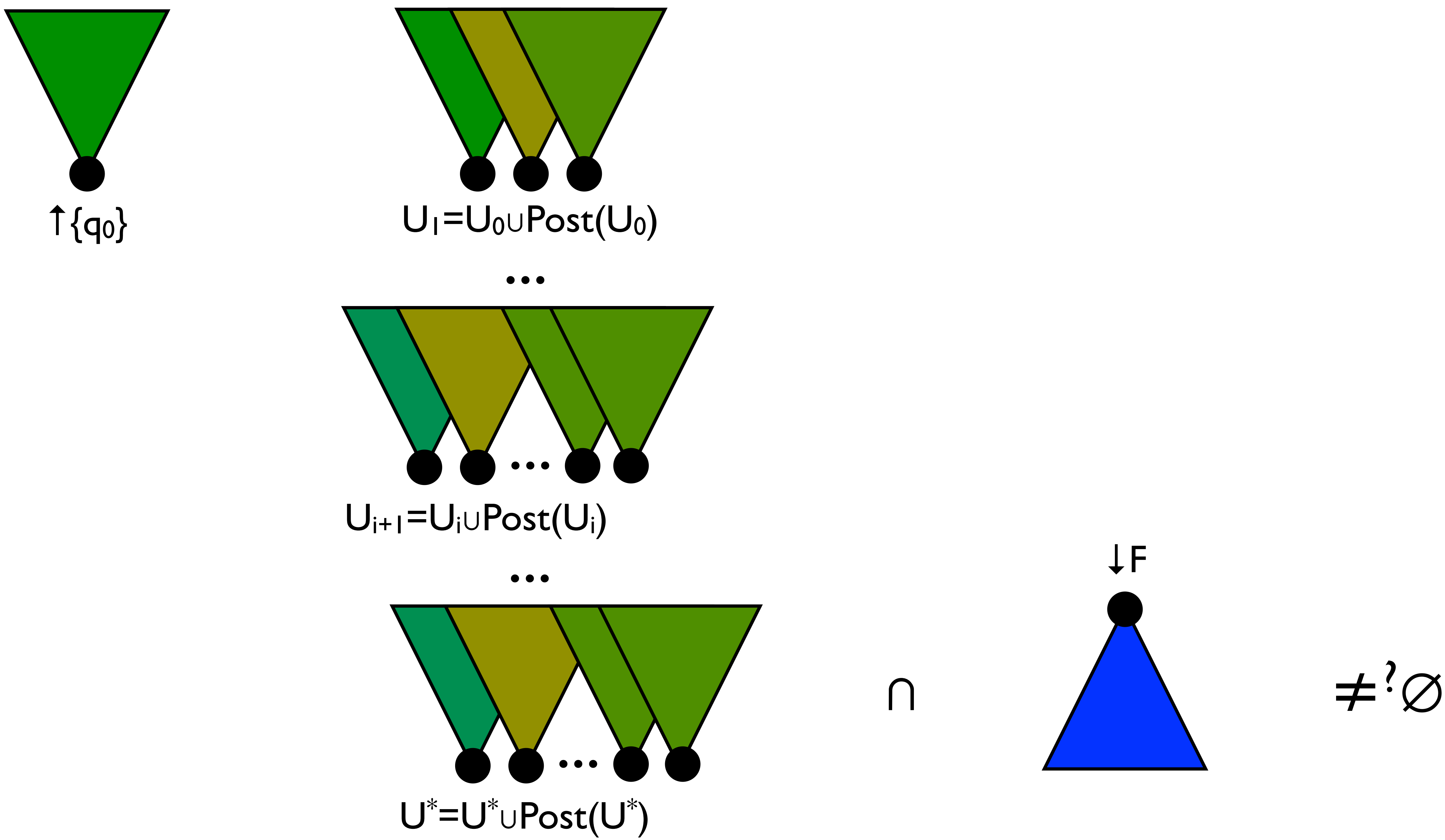
Init: sets containing initial states of A

Final: sets containing **no** accepting states of A

Forward analysis



Forward analysis



Forward analysis with antichains

$\uparrow\{q_0\}$

$U_1 = U_0 \cup Post(U_0)$

$\subseteq$-Upward-closed sets are canonically represented by their $\subseteq$-minimal elements

Very compact
Orders of magnitude faster than BDDs

$U_{i+1} = U_i \cup Post(U_i)$

…

$\downarrow F$

$\cap$

$\neq^{?} \varnothing$

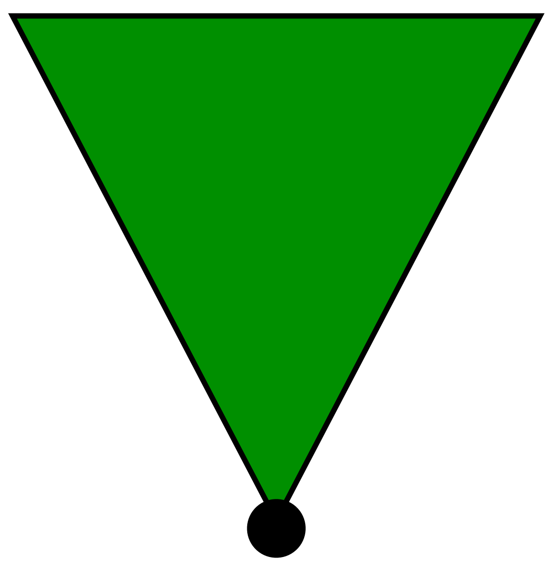
$U^* = U^* \cup Post(U^*)$

Discover post-fixpoint using SAT

- A set of sets $\mathbb{S} \subseteq 2^Q$ is a post-fixpoint of Post⟦A⟧ if:
  - $\{q_0\} \in \mathbb{S}$
  - $Post[\![A]\!](\mathbb{S}) \subseteq \mathbb{S}$
- Problem: find $\mathbb{S}$ such that $\mathbb{S} \cap F = \varnothing$
- Rely on the **antichain representation** of $\mathbb{S}$

Using SAT to synthesize $\mathbb{S}$

- Fix k the size of the antichain
- $X = \{ (q,i) \mid q \in Q \wedge 1 \leq i \leq k \}$
- any $v : X \rightarrow \{0,1\}$ represent an antichain



$\{ q \mid v(q,i)=1 \}$

Boolean encoding

- $\mathbb{S}$ is a post-fixpoint of Post⟦A⟧ and $\mathbb{S}$ does not intersect with ↓F
  - $\bigwedge_{i=1}^{i=k} \bigwedge_{\sigma \in \Sigma} \bigvee_{j=1}^{j=k} \bigwedge_{(q,i) \in X} (q,i) \rightarrow \bigwedge_{(q,j) | q \in \delta(q,\sigma)} (q,j)$
  - $(q_0, 1)$
  - $\bigwedge_{i=1}^{i=k} \bigvee_{q \in F} \neg(q,i)$

Boolean encoding

- 𝕊 is a post-fixpoint of Post⟦A⟧ and 𝕊 does not intersect with ↓F
  - $\bigwedge_{i=1}^{i=k} \bigwedge_{\sigma \in \Sigma} \bigvee_{j=1}^{j=k} \bigwedge_{(q,i) \in X} (q, i) \rightarrow \bigwedge_{(q,i)}$
  - $(q_0, 1)$
  - $\bigwedge^{i=k}$

Similar to template based inductive invariant generation using SMT solvers

Conclusion

- There are **several uses** of SAT solvers **beyond Bounded MC**
- SAT can be used **to help** SMC
- **UNSAT Core** are important and rich objects, useful for **abstraction refinements**
- **Interpolation** pushes the idea further (**no** more BDDs)
- Direct construction of **inductive invariants** can be useful too